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(Review Article)

Application of Trigonometric Functions to Analytical Solution of Certain Parial Differential Equations

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Abstract

This work presents an analytical study of several partial differential equations commonly used to model physical phenomena such as heat diffusion, wave propagation, and fluid flow. Emphasis is placed on the use of trigonometric functions to derive exact or synthetic solutions. The heat equations are then is examined using Fourier series and a complex-variable approach. The linearized Saint-Venant equations are then analyzed to describe shallow water wave propagation. The Burgers, in both inviscid and viscous forms, is used to illustrate nonlinear effects, damping, and shock formation. Finally, the Korteweg-de Vrie equation is discussed through its soliton solution, highlighting the balance between nonlinearity and dispersion. These results underline the importance of analytical and trigonometric methods in the modeling of thermal and hydraulic phenomena.

Keywords: Partial differential equations; Trigonometric functions; Heat equation; Saint-Venant equations; Burgers equation; Korteweg-de Vrie equation; Analytical solutions

1. Introduction

Partial differential equations (PDEs) play a central role in the mathematical modeling of many physical phenomena, such as heat diffusion, wave propagation, and fluid flow [1,2]. In this document, we focus on several classical PDEs-the heat equation, the Saint-Venant equations Partial, the Burgers equation, and the Korteweg-de Vries equation-for which simple analytical solutions can be constructed using trigonometric functions.

The use of sine and cosine functions naturally arises when the phenomena under study exhibit periodic or wave-like behavior. These functions form the basis of the Fourier method, which allows a solution to be represented as a superposition of elementary wave [4 ,5]. This approach is particularly well suited to the study of the heat equation, where it clearly describes the temporal evolution of temperature under given boundary conditions [5].

Furthemore, trigonometric functions also play an important role in the description of waves in fluid mechanics. The Saint-Venant equations, for example, model free-surface flows and the propagation of floods and inundation phenomena, especially in large river basins.

The objective of this work is to demonstrate, through several representative examples, how trigonometric functions can be used to construct and verify analytical solutions of PDEs. The study thus aims to establish a clear link between the mathematical tools taught in trigonometry and their concrete applications in physics and hydraulics.

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1.1. Heat equation

The heat equation describes the temporal evolution of temperature in a continuoius medium and constitutes a fundamental model for thermal diffusion phenomena [1,4]. It can be written as follows;

$$\boldsymbol{\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0} \qquad (1.1).$$

Where

$u(x,t)$ denotes the temperature at spatial position x and time t.

α denotes the thermal diffusivity of the medium, which characterizes the rate at which heat is transferred within the material.

1.1.1. Classical solution based on Fourier series

By considering a bar of finite length L with homogeneous boundary condition ($u(0,t) = u(L,t) = 0$), the solution of the heat equation can be expressed in the form of a Fourier series involving sine functions and decaying exponentials [4,5]. It is given by :

$$u(x,t) = A\ \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right)$$

with

$n = 1; 2; 3; \ldots$ denotes the mode number, corresponding to the number of thermal peaks along the bar;

A the initial maximum temperature amplitude;

$\sin\left(\frac{n\pi x}{L}\right)$ represents the spatial distribution of the temperature field along the bar;

$\exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right)$ characterizes the temporal decay of each mode, with higher modes (large n) or shorter bars (smaller L) exhibiting faster attenuation.

This combination of a sine function for the spatial dependence and an exponential function for the temporal evolution results from the homogeneous Dirichlet boundary conditions imposed at the ends of the bar. Since the temprature vanishes at the boundaries, sine functions naturally satisfy these conditions, as $\sin(0) = 0$. Furthermore, when the boundaries are maintained at zero temperature, thermal energy dissipates over time, leading to a gradual decay of the temperature field [5].

Why $u(x,t) = A\ \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right)$ is a synthetic solution of **(1.1) ?**

$$\frac{\partial u}{\partial t} = -\alpha \left(\frac{n\pi}{L}\right)^2 A \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right) ;$$

$$\frac{\partial u}{\partial x} = \frac{n\pi}{L} A \cos\left(\frac{n\pi}{L} x\right) \exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right) ;$$

$$\frac{\partial^2 u}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 A \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right) ;$$

As a result,

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = -\alpha \left(\frac{n\pi}{L}\right)^2 A \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right) + \alpha \left(\frac{n\pi}{L}\right)^2 A \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha \left(\frac{n\pi}{L}\right)^2 t\right) = 0$$

1.1.2. Non-classical solution obtained using the complex method

Here, we assume a semi-infinite rod, that is, $x > 0$.

We consider the one-dimensional heat equation with an oscillating boundary condition at the origin : $u(0,t) = A\cos(\omega t)$.

We seek a steady-state solution in the form of a complex wave, which allows us to use the tools of complex analysis [3,6]. It is expressed as :

$$\mathbf{u}(x,t) = \varphi(x)\exp(i\omega t) \qquad (1.2)$$

where $\varphi(x)$ is the complex amplitude depending on position. By substituting (1.2) into (1.1), we obtain;

$$\frac{\partial(\varphi(x)\exp(i\omega t))}{\partial t} = \alpha\frac{\partial^2(\varphi(x)\exp(i\omega t))}{\partial x^2} \Rightarrow i\omega\varphi(x)\exp(i\omega t) = \alpha\frac{d^2\varphi}{dx^2}\exp(i\omega t)$$

By simplifyting with $\exp(i\omega t)$, we obtain an ordinary differential equation (ODE) for $\varphi(x)$:

$$\frac{d^2\varphi}{dx^2} - \frac{i\omega}{\alpha}\varphi(x) = 0 \qquad (1.3)$$

This last equation is of the form $\varphi'' - k^2\varphi = 0$ with $k^2 = \frac{i\omega}{\alpha}$.

To extract k, we use the square root of the number i , so $k = \pm(1+i)\sqrt{\frac{\omega}{2\alpha}}$.

Therefore, $\varphi(x) = c_1\exp(-(\gamma + i\gamma)x) + c_2\exp((\gamma + i\gamma)x)$ with $\gamma = \sqrt{\frac{\omega}{2\alpha}}$.

Since the temperature must remain finite when x tends toward $+\infty$; we must impose$c_2 = 0$, otherwise the term would diverge.

And to obtain this, we use the boundary condition : $\varphi(0) = A \Rightarrow c_1 = A$.

The complex amplitude is therefore : $\varphi(x) = Aexp(-\gamma x)\exp(-i\gamma x)$.

Consequently, the real solution $u(x,t)$ is the real part of the complex solution $\mathbf{u}(x,t)$:

$$u(x,t) = Re(A\exp(-\gamma x)\exp(-i\gamma x)\exp(i\omega t)) = A\exp(-\gamma x)Re\big(\exp\big(i(\omega t - \gamma x)\big)\big)$$

However, according to Euler's identity [6], $\exp(i\theta) = \cos\theta + i\sin\theta$, so we obtain the final analytical solution :

$$u(x,t) = A\exp(-\gamma x)\cos(\omega t - \gamma x) = A\exp\left(-x\sqrt{\frac{\omega}{2\alpha}}\right)\cos\left(\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right).$$

1.2. Saint-Venant equation

To find a simple solution using trigonometric functions, we imagine that the water is almost calm with a constant depth H, and that we add a very small wave to it.

When disturbances are small, Saint-Venant's equations can be linearized to describe wave propagation in shallow water [7,8].

We obtain two equations that relate the small wave h (the change in height) and the velocity u :

$$\begin{cases} \frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0 \end{cases} \quad (2.1)$$

Where g is the force of gravity that pulls water downward.

$$\text{We consider} \begin{cases} u(x,t) = A\sqrt{\frac{g}{H}}\cos(kx - \omega t) \\ h(x,t) = H + A\ \cos(kx - \omega t) \end{cases} \quad \mathbf{(2.2)}$$

as a synthetic solution of **(2.1),** with

A : amplitude, i.e., the maximum height of the peak ;

k : represents the number of waves;

ω : is the frequency.

The choice of the cosine function is explained by the fact that we are looking for a solution that resembles a wave, i.e., a moving bump. But for this solution to work, the wave must move at a specific speed, denoted by c. This speed depends on the depth H and gravity g [9], so

$$c = \frac{\omega}{k} = \sqrt{gH}$$

This means that the deeper the wate (H is large), the faster the wave travels..

Furthermore, with **(2.2),** we obtain a perfect wave that moves to the right without ever changing shape. The cosine function creates the repetitive shape of the waves we see when we throw a stone into the water.

Let's verify that the system (2.2) does indeed form an analytical solution to (2.1)

.$\frac{\partial h}{\partial t} = Ak\sqrt{gH}\sin(kx - \omega t)$ and $\frac{\partial h}{\partial x} = -kA\sin(kx - \omega t)$;

$\frac{\partial u}{\partial t} = Akg\sin(kx - \omega t)$ and $\frac{\partial u}{\partial x} = -Ak\frac{\sqrt{gH}}{H}\sin(kx - \omega t)$;

As a result, $\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = Ak\sqrt{gH}\sin(kx - \omega t) - H\ Ak\frac{\sqrt{gH}}{H}\sin(kx - \omega t) = 0$

And also, $\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = Akg\sin(kx - \omega t) - gkA\sin(kx - \omega t) = 0.$

1.3. Burgers equation

The Burgers equations are mathematical equations used to describe how waves or fluids move [10]. They are used to model nonlinear flows, understand wave and shock formation, and test numerical methods used for more complex models such as the Navier-Stokes model or the Saint-Venant model.

1.3.1. One-dimensional Burgers equation without viscosity

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \mathbf{(3.1)}$$

This equation describes the formation of discontinuities, i.e., shocks [11]. It is used as a simple model to describe idealized shallow flows.

By imposing a sinusoidal initial condition$u(x,0) = A\sin(kx)$, The exact solution to (**3.2)** is given by :

$$u(x,t) = A\sin\big(k(x - ut)\big)$$

Where
A denotes the amplitude;
k is the wave number;
u is the propagation velocity.

Indeed, $\frac{\partial u}{\partial t} = -kAu\cos\big(k(x-ut)\big)$ and $\frac{\partial u}{\partial x} = kA\cos\big(k(x-ut)\big)$, therefore

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -kAu\cos\big(k(x-ut)\big) + ukA\cos\big(k(x-ut)\big) = 0.$

Hence the result.

1.3.2. One-dimensional Burgers equation with viscosity

We consider the following viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad \ldots\ldots. \qquad \mathbf{(3.3)}$$

Where $\nu > 0$ is the viscosity.

To solve this equation, mathematicians use a method called the Cole-Hopf transformation [12,13]. This involves replacing u with another function ϕ such that

$$u(x,t) = -2\nu\frac{1}{\phi}\frac{\partial \phi}{\partial x} \qquad \ldots\ldots\ldots \qquad \mathbf{(3.4)}$$

to arrive at the heat equation as a function of ϕ and ν [1,3]. However, the latter can be solved using trigonometric functions.

By choosing $\phi(x,t) = a_0 + a_1\exp\,(-\upsilon k^2 t)\cos(kx)$ and substituting it into **(3.4)**, we obtain an analytical solution to **(3.3)** given by :

$$u(x,t) = \frac{2\nu a_1 k exp(-\nu k^2 t)\sin(kx)}{a_0 + a_1 exp(-\nu k^2 t)\cos(kx)}$$

With
a_0 and a_1 are real numbers;
k is the wave frequency (a parameter that controls the distance between waves);

The exponential $exp(-\nu k^2 t)$ shows that the wave flattens out over time due to viscosity.

Indeed,

$$\frac{\partial u}{\partial t} = \frac{-2\nu^2 a_1 k^3 \exp(-\upsilon k^2 t)\, sin\, kx \quad (a_0 + a_1\exp(-\upsilon k^2 t)\cos kx)}{(a_0 + a_1\exp(-\upsilon k^2 t)\, cos\, kx)^2}$$

$$\frac{-2\nu a_1 k\exp(-\upsilon k^2 t)\, sin\, kx\,(-\nu a_1 k^2\exp(-\upsilon k^2 t)\, cos\, kx)}{(a_0 + a_1\exp(-\upsilon k^2 t)\, cos\, kx)^2}$$

$$= \frac{-2\nu^2 a_0 a_1 k^3\exp(-\upsilon k^2 t)\, sin\, kx - 2\nu^2 {a_1}^2 k^3\exp(-2\upsilon k^2 t)\cos kx\, sin\, kx}{(a_0 + a_1\exp(-\upsilon k^2 t)\, cos\, kx)^2}$$

$$+ \frac{2\nu^2 {a_1}^2 k^3\exp(-2\upsilon k^2 t)\cos kx\, sin\, kx}{(a_0 + a_1\exp(-\upsilon k^2 t)\, cos\, kx)^2}$$

$$= \frac{-2\nu^2 a_0 a_1 k^3\exp(-\upsilon k^2 t)\, sin\, kx}{(a_0 + a_1\exp(-\upsilon k^2 t)\, cos\, kx)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2va_1k^2 \exp(-vk^2t) \cos kx\,(a_0 + a_1 \exp(-vk^2t) \cos kx)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^2}$$

$$- \frac{2vka_1 \exp(-vk^2t) \sin kx(-ka_1 \exp(-vk^2t) \sin kx)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^2}$$

$$= \frac{2va_0a_1k^2 \exp(-vk^2t) \cos kx + 2va_1{}^2k^2 \exp(-2vk^2t)(\cos kx)^2 + 2va_1{}^2k^2 \exp(-2vk^2t)(\sin kx)^2}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^2}$$

$$= \frac{2va_0a_1k^2 \exp(-vk^2t) \cos kx + 2va_1{}^2k^2 \exp(-2vk^2t)((\cos kx)^2 + (\sin kx)^2)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^2}$$

$$= \frac{2va_0a_1k^2 \exp(-vk^2t) \cos kx + 2va_1{}^2k^2 \exp(-2vk^2t)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^2}$$

$$u\frac{\partial u}{\partial x} = \frac{2va_1k exp(-vk^2t) \sin(kx)}{a_0 + a_1 exp(-vk^2t) \cos(kx)} \times \frac{2va_0a_1k^2 \exp(-vk^2t) \cos kx + 2va_1{}^2k^2 \exp(-2vk^2t)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^2}$$

$$= \frac{4v^2a_0a_1{}^2k^3 exp(-2vk^2t) \sin(kx) \cos(kx) + 4v^2a_1{}^3k^3 exp(-2vk^2t) \sin(kx)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2va_0a_1k^3 \exp(-vk^2t) \sin kx(a_0 + a_1 \exp(-vk^2t) \cos kx)^2}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^4}$$

$$- \frac{2\left(2va_0a_1k^2 \exp(-vk^2t) \cos kx + 2va_1{}^2k^2 \exp(-2vk^2t)\right)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^4} \times$$

$$(-ka_1 \exp(-vk^2t) \sin kx)(a_0 + a_1 exp(-vk^2t) \cos(kx))$$

$$= \frac{-2va_0a_1k^3 \exp(-vk^2t) \sin kx(a_0 + a_1 \exp(-vk^2t) \cos kx)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3}$$

$$- \frac{2\left(2va_0a_1k^2 \exp(-vk^2t) \cos kx + 2va_1{}^2k^2 \exp(-2vk^2t)\right)(-ka_1 \exp(-vk^2t) \sin kx)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3}$$

$$= \frac{-2va_0{}^2a_1k^3 \exp(-vk^2t) \sin kx + 2va_0a_1{}^2k^3 \exp(-2vk^2t) \sin kx \cos kx + 4va_1{}^3k^3 \exp(-3vk^2t) \sin kx}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3}$$

$$v\frac{\partial^2 u}{\partial x^2} = \frac{-2v^2a_0{}^2a_1k^3 \exp(-vk^2t) \sin kx + 2v^2a_0a_1{}^2k^3 \exp(-2vk^2t) \sin kx \cos kx}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3}$$

$$+ \frac{4v^2a_1{}^3k^3 \exp(-3vk^2t) \sin kx}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{-2v^2a_0a_1k^3 \exp(-vk^2t) \sin kx}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^2} +$$

$$\frac{4v^2a_0a_1{}^2k^3 exp(-2vk^2t) \sin(kx) \cos(kx) + 4v^2a_1{}^3k^3 exp(-2vk^2t) \sin(kx)}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3}$$

$$= \frac{-2v^2a_0a_1k^3 \exp(-vk^2t) \sin kx\,(a_0 + a_1 exp(-vk^2t) \cos(kx))}{(a_0 + a_1 \exp(-vk^2t) \cos kx)^3} +$$

$$\frac{4\nu^2 a_0 {a_1}^2 k^3 \exp(-2\nu k^2 t)\sin(kx)\cos(kx) + 4\nu^2 {a_1}^3 k^3 \exp(-2\nu k^2 t)\sin(kx)}{(a_0 + a_1 \exp(-\upsilon k^2 t)\, cos\, kx)^3}$$

$$= \frac{-2\nu^2 {a_0}^2 a_1 k^3 \exp(-\upsilon k^2 t)\, sin\, kx + 2\nu^2 a_0 {a_1}^2 k^3 \exp(-2\upsilon k^2 t)\sin kx\, cos\, kx}{(a_0 + a_1 \exp(-\upsilon k^2 t)\, cos\, kx)^3} + \frac{4\nu^2 {a_1}^3 k^3 \exp(-3\upsilon k^2 t)\sin kx}{(a_0 + a_1 \exp(-\upsilon k^2 t)\, cos\, kx)^3}$$

From result,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

1.4. Korteweg-de Vrie (KdV) equation

This equation is widely used in studies of tidal waves and waves in channels. It describes how a water bump can travel long distances without changing shape [14]. However, the Kortewerg-de Vrie equation admits a very special solution called a soliton.

A soliton is a single wave (a single crest) that travels at a constant speed without ever changing shape, even after encountering another wave [15].

It is generally given by :

$$\frac{\partial h}{\partial t} + 6h\frac{\partial h}{\partial x} + \frac{\partial^3 h}{\partial x^3} = 0 \quad \mathbf{(4.1)}$$

Given the characteristics of a soliton, to construct an analytical solution for (**4.1),** we imagine a wave moving to the right at a constant speed c and use the hyperbolic secant function, which resembles a bell curve.

Thus, an analytical solution for **(4.1)** is given by :

$$h(x,t) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$

Indeed, $\frac{\partial h}{\partial t} = 2 \times \frac{c}{2}\left(\frac{c\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)\text{sech}\left(\frac{\sqrt{c}}{2}(x-ct)\right)$

$$= \frac{c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$

$$\frac{\partial h}{\partial x} = -\frac{c\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$

$$6h\frac{\partial h}{\partial x} = 6\frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\left(-\frac{c\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)$$

$$= -\frac{3c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$

$$\frac{\partial^2 h}{\partial x^2} = -\frac{c\sqrt{c}}{2}\left[\frac{\sqrt{c}}{2}\left(1-\tanh^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)+\sqrt{c}\tanh^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right]$$

$$= -\frac{c^2}{2}\left(\frac{1}{2}\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)-\tanh^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)$$

$$= -\frac{c^2}{4}\left(\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)-2\tanh^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)$$

$$\frac{\partial^3 h}{\partial x^3} = -\frac{c^2}{4}\left[4\frac{\sqrt{c}}{2}\left(-\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)\right.$$
$$-2\left(\sqrt{c}\left(1-\tanh^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right.$$
$$\left.\left.-\sqrt{c}\tanh^3\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)\right]$$

$$= -\frac{c^2\sqrt{c}}{2}\left(-\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)-\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right.$$
$$\left.+2\tanh^3\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)$$

$$= -\frac{c^2\sqrt{c}}{2}\left(-\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)-\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right.$$
$$\left.+2\left(1-\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)$$

$$= -\frac{c^2\sqrt{c}}{2}\left(\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)-3\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)$$

$$\frac{\partial h}{\partial t}+6h\frac{\partial h}{\partial x}+\frac{\partial^3 h}{\partial x^3}$$
$$=\frac{c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)-\frac{3c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$
$$-\frac{c^2\sqrt{c}}{2}\left(\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)-3\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)$$

$$=\frac{c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)-\frac{3c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$
$$-\frac{c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)+\frac{3c^2\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)\text{sech}^4\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$

From where, $h(x,t)=\frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$ verifies the equation $\frac{\partial h}{\partial t}+6h\frac{\partial h}{\partial x}+\frac{\partial^3 h}{\partial x^3}=0$.

2. Conclusion

This work has highlighted the fundamental role of trigonometric functions in the construction of analytical solutions to several classical partial differential equations. Through the study of the heat equation, we have shown how Fourier series can be used to describe thermal diffusion and the gradual cooling of a medium subjected to simple boundary condtitions. The approach based on complex functions has also illustrated the contribution of complex analysis to the treatment of oscillatory regimes.

The analysis of the Saint-Venant equations has made it possible to gain a deeper understanding of shallow water wave propagation and to relate wave speed to essential physical parameters such as gravity and the mean flow depth. These results provide an important theoretical basis for the modeling of free-surface flows and the study of flood phenomena.

The Burgers equation has then been used as a simplified model to illustrate the effects of nonlinearity and viscosity, particular shock formation and wave damping. Finally, the Korteweg-de Vrie equation introduced the concept of solitons, highlighting the balance between nonlinearity and dispersion that allows a wave to propagate without deformation.

In coclusion, this study demonstrates that trigonometric and related function constitute powerful mathematical tools for understanding and modeling real physical phenomena. It provides a solid theoretical foundation dor the study of more complex models and for the further development of numerical methods applied to thermal and hydraulic problems.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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