World Journal of Advanced Research and Reviews
eISSN: 2581-9615 CODEN (USA): WJARAI
Cross Ref DOI: 10.30574/wjarr
Journal homepage: https://wjarr.com/

WJARR World Journal of Advanced Research and Reviews

(Research Article)

Check for updates

Theoretical and numerical study of the critical threshold of linear stability for the flow of a weakly viscoelastic fluid in a cylindrical pipe with a horizontal axis

Ibrahima Kama *, Mamadou Yacine Ba, Alpha Malick Ndiaye and Cheikh Mbow

Department of physic, Faculty of sciences and technology, University Cheikh Anta Diop of Dakar, Dakar, Senegal.

World Journal of Advanced Research and Reviews, 2025, 28(03), 945-953

Publication history: Received on 05 November 2025; revised on 12 December 2025; accepted on 15 December 2025

Article DOI: https://doi.org/10.30574/wjarr.2025.28.3.4145

Abstract

In this paper, we seek to determine the critical Reynolds number of a viscoelastic fluid flowing in a cylindrical pipe with a horizontal axis. The problem obtained is a generalized eigenvalue problem $Av = \chi Bv$. A Gauss-Lobatto-Tchebyshev method was adopted to discretize this equation and the QZ algorithm combined with the Newton-Raphson method was used to determine this critical value of the Reynolds number. It is obtained by searching for two successive and very close values for which correspond two eigenvalues whose maximum real parts are respectively negative and positive. In other words, the critical value is the smallest value of the Reynolds number for which instability occurs. The code for performing this calculation was written in FORTRAN.

The flow is stable if all the real parts of the eigenvalues obtained are negative and unstable if only one of these values is positive.

Keywords: Viscoelastic fluids; Linear instability; Petrov-Galerkin; Generalized eigenvalue problem; Algorithm QZ; Critical Reynolds' number

1. Introduction

The critical value of the Reynolds number for a flow in a cylindrical pipe varies according to the amplitude of the disturbances. Thus, it has been proven that this critical value is a decreasing function of the amplitude of the disturbance.

The literature on the study or determination of the value of the critical Reynolds number includes two parts: the experimental part and the numerical part. Regarding the experimental part, we can list the work of Osborne Reynolds carried out in 1883. According to his results, this critical value depends on the amplitude of the disturbance. He thus concludes that the flow of a Newtonian fluid in a pipe is linearly stable regardless of the amplitude of the disturbances if $Re < 2000$ and linearly unstable for $Re > 13000$. In the same vein, Pfenninger showed that the lower limit beyond which a flow of a Newtonian fluid in a pipe is linearly unstable is $Re = 10^5$.

Numerical studies have shown that the flow of a Newtonian fluid in a pipe is linearly stable. Examples include the work of **Meseguer and Trefethen 2001 [1]** or **Kama et al. 2018 [2]**. For viscoelastic fluids, we were unable to find any literature.

1.1. Equations and mathematical model

We study the flow of a viscoelastic fluid along a cylinder of circular section of horizontal axis (Oz). Such a flow can be described by a cylindrical coordinate system r, θ, z.

* Corresponding author: Ibrahima Kama

The flow of a viscoelastic fluid can be described using three main equations: the momentum conservation equation, the mass conservation equation (or continuity equation), and the behavioral equation of the extra-constraint (***Chupin 2003[3] and Oldroyd 1950 [4])***. We consider the flow of an incompressible viscoelastic fluid with dynamic viscosity μ and density ρ driven by an extremal constant axial pressure gradient.

$$\begin{cases} \rho\left(\dfrac{\partial\vec{\boldsymbol{u}}}{\partial t}+(\vec{\boldsymbol{u}}.\nabla)\vec{\boldsymbol{u}}\right)=-\nabla P+\vec{f}+div\{2\eta(1-\omega)\mathrm{D}(\vec{\boldsymbol{u}})\}\ +\mathrm{div}\sigma \\ \dfrac{\partial\rho}{\partial t}+div(\rho\vec{\boldsymbol{u}})=0 \\ \dfrac{D\sigma}{Dt}+\dfrac{\sigma}{\lambda}=\dfrac{2\eta\omega}{\lambda}D(\vec{\boldsymbol{u}}) \end{cases} \quad (1)$$

By subtracting from this flow the basic flow, these equations in dimensionless form are respectively:

$$\frac{\partial\vec{\boldsymbol{u}}'}{\partial t}=-(\overrightarrow{W_b}.\boldsymbol{\nabla})\vec{\boldsymbol{u}}'-(\vec{\boldsymbol{u}}'.\boldsymbol{\nabla})\overrightarrow{W_b}-\boldsymbol{\nabla}p'+\frac{(1-\omega)}{R_e}\Delta\vec{\boldsymbol{u}}'+\mathbf{div}\sigma' \quad (2)$$

$$div\vec{\boldsymbol{u}}'=0 \quad (3)$$

$$\frac{\partial\sigma'}{\partial\mathrm{t}}+u'\frac{\partial\sigma_{\mathrm{b}}}{\partial r}+\mathrm{W_b}\frac{\partial\sigma'}{\partial z}+g_a(\vec{\boldsymbol{u}}',\sigma_{\mathrm{b}})+g_a(\overrightarrow{W_b},\sigma')+\frac{\sigma'}{W_e}=\frac{2\omega}{R_eW_e}D(\vec{\boldsymbol{u}}') \quad (4)$$

Considering in addition that the elastic contribution of the extra stress is a perturbation of the Newtonian one σ'_s which is equal to $\frac{2\omega}{R_e}D(\vec{\boldsymbol{u}}')$, it comes:

$$\frac{\partial\sigma'_e}{\partial t}+W_b^s\frac{\partial\sigma'_e}{\partial z}+\frac{\sigma'_e}{W_e}+\sigma'_e\nabla\left(\overrightarrow{W_b^s}\right)=-\frac{2\omega}{R_e}\left\{W_b^e\partial_zD(\vec{u})+D(\vec{u})\nabla\left(\overrightarrow{W_b^e}\right)\right\} \quad (5)$$

By $\vec{u}$, we define the adimensional velocity field of the fluid whose radial, azimuth and axial components are given in the order by the triple (u,v,w), σ is an additional term called extra stress and ω is the parameter of delay defined by

$$\omega=1-\frac{\lambda'}{\lambda} \quad \ldots\ldots \quad (6)$$

The system is closed with the following boundary and initial conditions:

$$u=0\quad si\ \ r=1 \quad \ldots\ldots \quad (7)$$

$$u=u_0\quad \text{à}\ \ t=0 \quad \ldots\ldots \quad (8)$$

Where $\frac{RW_0}{\nu}=Re$ is defined as the Reynolds number with $\nu=\frac{\eta}{\rho}$ the kinematic viscosity of the fluid, $We=\frac{\lambda W_0}{R}$ is the Weissenberg's number.

2. Procedure and numerical bases

We consider that the variation of the perturbation of the fields of velocity, pressure and extra stress is periodic along the azimuthal and axial directions. These considerations make it possible to approximate u in the following form Meseguer & Trefethen 2001[1]:

$$\boldsymbol{u_s}(r,\theta,z;t)=\sum_{l=-L}^{L}\sum_{n=-N}^{N}\sum_{m=0}^{M}\alpha_{mnl}(t)\phi_{mnl}(r)e^{i(n\theta+l\lambda_0z)} \quad (12)$$

The continuity equation leads to a linear dependence between the three components of $\mathrm{u}(r,\theta,z)$ leading to a system with two degrees of freedom. Therefore, we note:

$$u_s(r,\theta,z;t) = \sum_{l=-L}^{L}\sum_{n=-N}^{N}\sum_{m=0}^{M}\left[\alpha_{mnl}^{(j)}(t)\phi_{mnl}^{(j)}(r)\right]e^{i(n\theta+l\lambda_0 z)} \quad (13)$$

For $j = (1,2)$ the substitution of (13) in the equation governing the problem result in a system of ordinary differential equations of coefficients $\alpha_m^{(j)}(t)$. This substitution is followed by a projection based on the scalar product:

$$(f,g) = \int_{-1}^{1} w(z)f^{*}.gdz \qquad \dots\dots\dots \qquad (14)$$

Where f^{*} designates the conjugated function of f.

The basic functions will be chosen so that integrand is even, which will result in:

$$\int_{-1}^{1} G(r)dr = 2\int_{0}^{1} G(r)dr \qquad \dots\dots \qquad (15)$$

The basic functions are also chosen so that the projection of the pressure term is zero. For this purpose, functions based on Chebyshev polynomial were considered. This choice imposes two essential criteria. The first consists of choosing weights associated with Chebyshev polynomials, which makes it easier to calculate the scalar product. The second criterion relates to the approximation of the integral by a Gauss- Chebyshev- Lobatto quadrature of the form:

$$(f,g) = \int_{0}^{1} w(r)f^{*}.gdr \approx \sum_{j=0}^{M} w(r_j)f^{*}(r_j)g(r_i) \quad (16)$$

$$f = \phi_{m,n,l}^{(1,2)} \ et\ g = \psi_{m,n,l}^{(1,2)} \qquad (17)$$

We define,

$$h_m(r) = (1-r^2)T_{2m} \quad (18)\,, \quad g_m(r) = (1-r^2)^2 T_{2m} \quad (19)$$

T_{2m} is the Chebyshev polynomial defined by $T_{2m} = \cos(2m\theta)$

The basic functions and tests were proposed by Leonard & Wray 1982 [5] then used by Meseguer & Trefethen 2003 [6] and are defined as follows:

2.1. Basic functions

2.1.1. 1st case : $\boldsymbol{n = 0}$

$$\phi_{m,l,n}^{(1)} = \begin{pmatrix} 0 \\ rh_m(r) \\ 0 \end{pmatrix}$$

$$\phi_{m,l,n}^{(2)} = \begin{pmatrix} ik_0 lrg_m(r) \\ 0 \\ \begin{cases} D_+[rg_m(r)] & si\ l \neq 0 \\ h_m(r)\ si\ l = 0 \end{cases} \end{pmatrix} \quad (20)$$

2.1.2. 2nd case : $n \neq 0$

$$\phi^{(1)}_{m,l,n} = \begin{pmatrix} -inr^{\delta-1}g_m(r) \\ D[r^{\delta}g_m(r)] \\ 0 \end{pmatrix}$$

$$\phi^{(2)}_{m,l,n} = \begin{pmatrix} 0 \\ -ik_0 lr^{\delta+1}h_m(r) \\ inr^{\delta}h_m(r) \end{pmatrix} \quad (21)$$

$$\delta = \begin{cases} 2 \ if \ n \ is \ even \\ 1 \ if \ n \ is \ odd \end{cases} \quad (22)$$

2.2. Test functions

2.2.1. 1st case : $\boldsymbol{n = 0}$

$$\psi^{(1)}_{m,l,n} = \begin{pmatrix} 0 \\ h_m(r) \\ 0 \end{pmatrix} \frac{1}{\sqrt{1-r^2}} \quad (23)$$

$$\psi^{(2)}_{m,l,n} = \begin{pmatrix} -ik_0 lr^2 g_m(r) \\ 0 \\ \begin{cases} D_+[r^2 g_m(r) + r^3 h_m(r)] \quad si \ l \neq 0 \\ rh_m(r) \ \ si \ l = 0 \end{cases} \end{pmatrix} \frac{1}{\sqrt{1-r^2}} \quad (24)$$

2.2.2. 2nd case : $\boldsymbol{n \neq 0}$

$$\psi^{(1)}_{m,l,n} = \begin{pmatrix} -inr^{\beta}g_m(r) \\ D[r^{\beta+1}g_m(r) + r^{\beta+2}h_m(r)] \\ 0 \end{pmatrix} \frac{1}{\sqrt{1-r^2}} \quad (25)$$

$$\psi^{(2)}_{m,l,n} = \begin{pmatrix} 0 \\ ik_0 lr^{\beta+2}h_m(r) \\ \begin{cases} [-inr^{\beta+1}h_m(r)] \quad si \ l \neq 0 \\ r^{1-\beta}h_m(r) \ \ si \ l = 0 \end{cases} \end{pmatrix} \frac{1}{\sqrt{1-r^2}} \quad (26)$$

$$\beta = \begin{cases} 0 \ if \ n \ is \ even \\ 1 \ if \ n \ is \ odd \end{cases} \quad (27)$$

Using the Fourier representation, the continuity equation becomes

$$Du(r) + \frac{in}{r}v(r) + i\lambda_0 w(r) = 0 \quad (28)$$

$$D = \frac{d}{dr} \quad (29)$$

3. Numerical implementation

Let

$$R_u = -\frac{\partial \vec{u}}{\partial t} - (\vec{u}.\vec{\nabla})\vec{u} - \vec{\nabla}\mathrm{p} + \frac{\omega}{\mathrm{R_e}}\Delta\vec{\mathrm{u}} - W_e\vec{\nabla}.\left(\frac{D\sigma}{Dt}\right) \quad (26)$$

Replacing $\vec{\mathrm{u}}$ by the basic functions and projecting on the basis of the test functions, it comes:

$$(R_u \, , \psi) = 0 \quad (27)$$

The numerical method chosen is adapted from work in the early 1980 by Leonard and Wray ***[5]***.

With the Petrov-Galerkin procedure (***Meseguer & Trefethen 2000*** **[7]) and (*Meseguer & Melliboveský 2007 [8])***, we obtain:

$$\begin{pmatrix} (\phi^1_{m,n,l},\psi^1_{m,n,l}) & (\phi^2_{m,n,l},\psi^1_{m,n,l}) \\ (\phi^1_{m,n,l},\psi^2_{m,n,l}) & (\phi^2_{m,n,l},\psi^2_{m,n,l}) \end{pmatrix} \begin{pmatrix} \dot{\alpha}^1_{m,n,l} \\ \dot{\alpha}^2_{m,n,l} \end{pmatrix} + \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} + W_e \begin{pmatrix} d^1 \\ d^2 \end{pmatrix} = \frac{1}{R_e} \begin{pmatrix} (\Delta\phi^1_{m,n,l},\psi^1_{m,n,l}) & (\Delta\phi^2_{m,n,l},\psi^1_{m,n,l}) \\ (\Delta\phi^1_{m,n,l},\psi^2_{m,n,l}) & (\Delta\phi^2_{m,n,l},\psi^2_{m,n,l}) \end{pmatrix} \begin{pmatrix} \alpha^1_{m,n,l} \\ \alpha^2_{m,n,l} \end{pmatrix} \quad (28)$$

where

$$c^{1,2} = \left((\vec{u}.\vec{\nabla})\vec{u}, \psi^{1,2}_{m,n,l}\right) \; and \; d^{1,2} = \left(\vec{\nabla}.\left(\frac{D\sigma}{Dt}\right), \psi^{1,2}_{m,n,l}\right) \quad (29)$$

The matrices $c^{1,2}$ and $d^{1,2}$ derive from the projection of the nonlinear terms and can be calculated with a pseudo-spectral method by Fast Fourier Transform (FFT).

$$A = \frac{1}{R_e} \begin{pmatrix} (\Delta\phi^1_{m,n,l},\psi^1_{m,n,l}) & (\Delta\phi^2_{m,n,l},\psi^1_{m,n,l}) \\ (\Delta\phi^1_{m,n,l},\psi^2_{m,n,l}) & (\Delta\phi^2_{m,n,l},\psi^2_{m,n,l}) \end{pmatrix} \begin{pmatrix} \alpha^1_{m,n,l} \\ \alpha^2_{m,n,l} \end{pmatrix} + \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} + \begin{pmatrix} d^1 \\ d^2 \end{pmatrix}$$

$$B = \begin{pmatrix} (\phi^1_{m,n,l},\psi^1_{m,n,l}) & (\phi^2_{m,n,l},\psi^1_{m,n,l}) \\ (\phi^1_{m,n,l},\psi^2_{m,n,l}) & (\phi^2_{m,n,l},\psi^2_{m,n,l}) \end{pmatrix}$$

The problem obtained after projection is a problem with the generalized eigenvalues that we will solve numerically with QZ algorithm *J. P. Berlioz [9]*. To implement this method, we will use Housholder's unitary reflection matrices and Givens rotation matrices (*A. Quarteroni, R. Sacco and F. Saleri* [10]) and *(L. Amodei and J.P. Dedieu [11])*.

4. Results and discussions

We will treat this problem according to four cases that are:

Case of one-dimensional disturbance (n=0, k0=0)

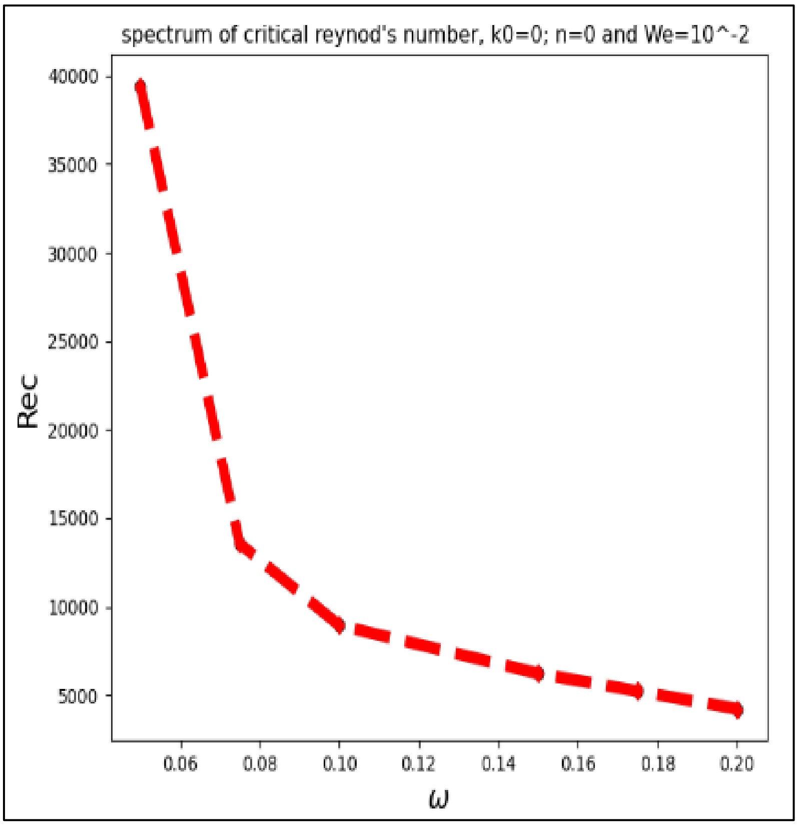

Figure 1 Spectrum of critical Reynolds' number as a function of the delay parameter ω with $k_0 = 0.$;n=0 and $We = 10^{-2}$

Case of homogeneous Disturbance (n≠0, k0=0)

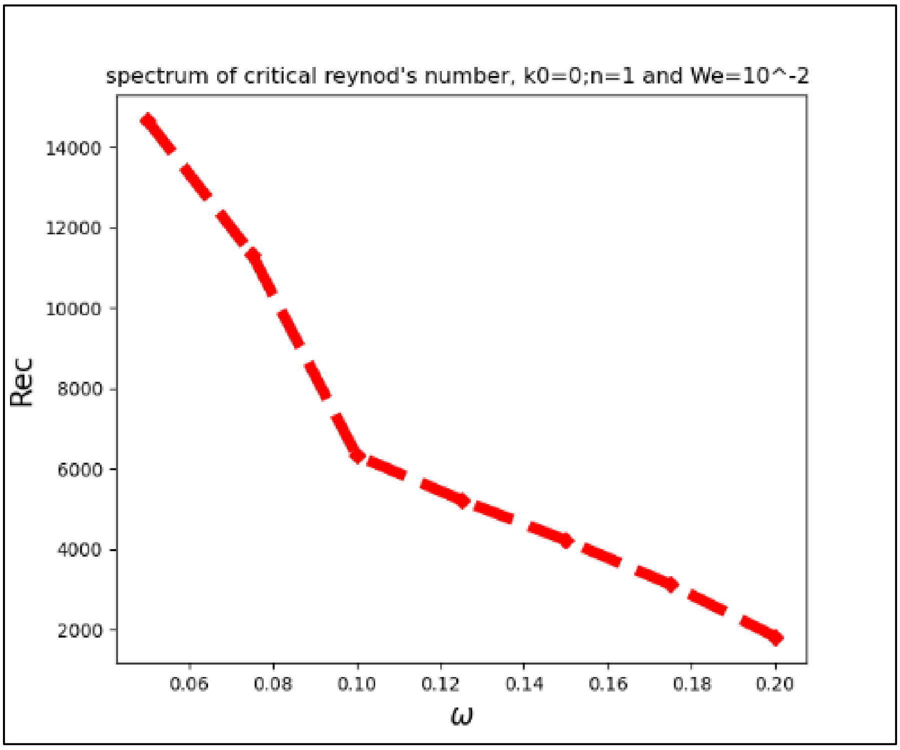

Figure 2 Spectrum of critical Reynolds' number as a function of the delay parameter ω with $k_0 = 0.; n = 1$ and $We = 10^{-2}$

Case of Axisymmetric Disturbance (n=0, k0 ≠0)

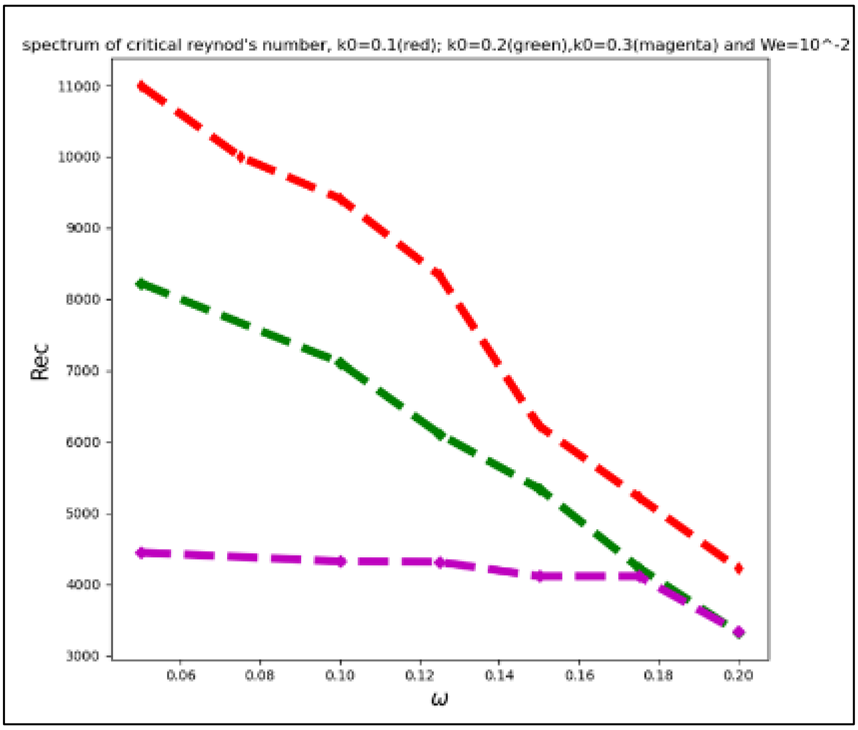

Figure 3 Spectrum of critical Reynolds' number as a function of the delay parameter ω with $n = 0$; $k_0 = 0.1$ *(red),* ; $k_0 = 0.2$ *(green),* ; $k_0 = 0.3$ *(magenta)* and $We = 10^{-2}$

Case of Three-dimensional disturbance (n≠0, k0≠0)

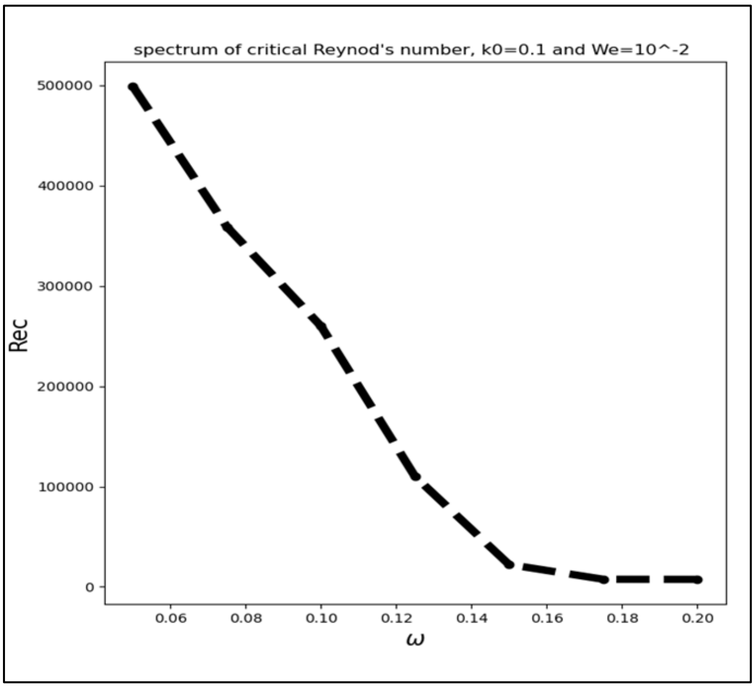

Figure 4 Spectrum of critical Reynolds' number as a function of the delay parameter ω with $n = 1$; $k_0 = 0.1$ and $We = 10^{-2}$

It is generally observed that the critical Reynolds number decreases with the delay parameter. This decrease depends on the axial and azimuthal wavenumbers. It is more pronounced when the axial wavenumber is low. The critical

Reynolds number decreases as the axial wavenumber increases. That is to say, for two different axial wavenumbers and for the same value of the delay parameter, the critical Reynolds number is reached more frequently with the higher value.

5. Conclusion

Depending on the nature of the disturbance and the flow parameters considered in this study, we find that the smallest Reynolds number value below which any flow is stable is 1824 and the Reynolds number value above which any flow is unstable is 5.10^5. However, it is observed that above a certain value of the axial wavenumber, the critical Reynolds number remains almost invariant with respect to the delay parameter. This means that for certain values of the axial wavenumber, the delay parameter (considered small) no longer influences stability.

The following facts can explain these results: - Elasticity is the driving force behind the instability. It should be noted that for zero elasticity (Newtonian fluid), no instability is observed. - At low axial wavenumbers or long wavelengths, diffusive effects are significant. These diffusive effects, having a stabilizing aspect, require even greater inertia and therefore a higher Reynolds number to trigger instabilities.

Nomenclature

- Greek letters
  - θ : Azimuthal coordinate
  - λ : Relaxation time of the fluid
  - λ' : delay time
  - ν: Kinematic viscosity of the fluid
  - ρ : Density of the fluid
  - σ: Tensor of extra stress
  - σ': Disturbance of extra-stress
  - σ^s: Newtonian contribution of the disturbance of the extra-stress
  - σ^e: Elastic contribution of the disturbance of the extra-stress
  - η : Dynamic viscosity of the fluid
  - ϕ: Basis function
  - ψ: Test function
  - ω: delay parameter
- Latin letters
  - C: eigenvalue
  - C_i:imaginary part of C
  - C_r:real part of C
  - C_m: real part of the most unstable eigenvalue
  - f: density force
  - l: mode axial
  - k0 : Axial wave number
  - n: mode azimutal
  - r : radial coordinate
  - R: radius of the cylinder
  - Re: Reynolds number
  - Rec : Critical Reynolds' number
  - W_0 : Maximum speed of the base flow, it has for direction that of the axis of the cylinder.
  - We: Weissenberg's number
  - Wb: Axial velocity of the base flow
  - W_b^s: Newtonian contribution of basic flow velocity
  - W_b^e: Elastic contribution of basic flow velocity
  - Z: axial coordinate

6. Conclusion

A study of the critical Reynolds number as a function of the delay parameter was conducted. This study revealed that the critical Reynolds number decreases with the delay parameter for certain wavenumber values. However, it was observed that for some wavenumber values, for example, $k_0 = 0.3$, the critical Reynolds number is almost constant. This finding has sparked considerable interest for us and leads us to ask some questions that seem legitimate.

Is there a critical axial wavenumber for a given Reynolds number?

Is there a critical delay parameter for a given wavenumber and Reynolds number? Beyond these questions, one might be tempted to study the influence of the azimuthal wavenumber on stability or, at best, conduct a comparative study between diffusive and elastic effects.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Statement of ethical approval

The authors declare that all applicable ethical standards have been followed during the conduct of this research.

References

[1] A spectral Petrov-Galerkin formulation for pipe flow II: Nonlinear transitional stages

[2] I. kama et al. "Effects of the Reynolds' and Weissenberg's numbers on the stability of linear pipe flow of viscous fluid". Journal of Modern Mechanical Engeneering, 2018,08, 264-277 http://www.scirp.org/journal/mme

[3] L. Chupin, Doctoral thesis, MAB, University of Bordeaux 1, Contribution to the study of viscoelastic fluid mixtures, 2006

[4] Oldroyd, J. (1950). On the formation of rheological equation of state. London: Proc. R. Soc A200.

[5] Leonard, A. and Wray, A. (1982) A New Numerical Method for the Simulation of Three-Dimensional Flow in a Pipe. In: Krause, E., Ed., Proceedings of the 8th International Conference on Numerical Methods in Fluid Dynamics on Numerical Methods in Fluid Dynamics, Springer, Berlin, 335-342. https://doi.org/10.1007/3-540-11948-5_40

[6] Meseguer, A. and Trefethen, L.N. (2003) Linearized Pipe Flow to Reynolds Number 10^7 . Journal of Computational Physics, 186,178-197.https://doi.org/10.1016/S0021-9991(03)00029-9

[7] Meseguer, A. and Trefethen, L.N. (2000) A Spectral Petrov-Galerkin Formulation for Pipe Flow (I): Linear Stability and Transient Growth. Numerical Analysis Group, Computing Lab, Oxford University, Rep. 00/18.

[8] Meseguer, A. and Malinovsky, F. (2007) On a solenoidal Fourier-Chebyshev Spectral Method for Stability Analysis of the Hagen-Poiseuille Flow. Applied Numerical Mathematics, 57, 920-938. https://doi.org/10.1016/j.apnum.2006.09.002

[9] Berlioz, J.P. (1979) A propos de l'algorithme QZ. RAIRO-Analyse Numérique, 13,21-30. https://doi.org/10.1051/m2an/1979130100211

[10] Quarteroni, A., Sacco, R. and Saleri, F. (2004) Numerical Methods: Algorithms, analysis and applications. Springer-Verlag, Italia, Milano.

[11] Amodei, L. and Dedieu, J.P. (2008) Numerical Matrix Analysis: Course and detailed solved exercises. Dunod, Paris 2008, SMAI (Society for Applied and Industrial Mathematics