World Journal of Advanced Research and Reviews

eISSN: 2581-9615 CODEN (USA): WJARAI

Cross Ref DOI: 10.30574/wjarr

Journal homepage: https://wjarr.com/

(Research Article)

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Covering Systems in Algebraic Domains: Analogues in Number Fields and Function Fields

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World Journal of Advanced Research and Reviews, 2025, 28(03), 877-889

Publication history: Received 04 November 2025; revised on 10 December 2025; accepted on 13 December 2025

Article DOI: https://doi.org/10.30574/wjarr.2025.28.3.4143

Abstract

Classical covering systems, introduced by Erdo's, consist of congruences a_i (mod n_i) whose union covers all integers. Despite extensive work on their structural and extremal properties, little is known about analogues in algebraic settings. In this paper, we develop a unified framework for covering systems over algebraic domains, focusing on the ring of integers $\mathcal{O}_K$ of a number field and the polynomial ring $F_q[x]$. We define algebraic covering systems in both environments and establish necessary norm and degree-based conditions for full coverage and demonstrate that restricted families of ideals or polynomial moduli cannot yield coverings unless their reciprocal norm sums exceed explicit thresholds. We further provide structural examples, and counterexamples illustrating how factorization patterns, prime splitting, and residue structure influence covering behavior. Our results show that polynomial rings admit sharper and more uniform obstruction criteria than number fields, while number-field coverings exhibit arithmetic constraints governed by prime ideal decomposition.

Keywords: Covering Systems; Modulus; Integers; Ideals; Sieve Theory

1. Introduction

A *covering system* is a finite family of congruences

$$m \equiv a_i \pmod{n_i},$$

such that every integer satisfies at least one of them. If moduli $n_1, n_2, \ldots, n_k$ are pairwise distinct, the system is called a *distinct covering system*. Covering systems were introduced by Erdo's [3] and have since played a central role in combinatorial and analytic number theory. Early constructions by Moser [10] and Selfridge [17] demonstrated the surprising variety of covering systems and inspired deeper investigation into their extremal structure. A long-standing problem posed by Erdo's asked whether distinct covering systems could have arbitrarily large least modulus. This problem was resolved in a breakthrough by Hough [8], who used probabilistic and entropy-based tools to show that there exists an absolute constant M such that no distinct covering system may have all moduli exceeding M.

A parallel development due to [6] introduced a sieve-theoretic framework for proving nonexistence of covering systems. Their results show that if a set of moduli is too "sparse" in the sense that the reciprocal sum $\sum \frac{1}{n_i}$ is too small, then the corresponding congruence classes necessarily leave a positive density of integers uncovered [13].

Despite much progress in the classical setting, relatively little attention has been given to *algebraic analogues* of covering systems in other arithmetic rings. Two natural and important generalizations are: the ring of integers $\mathcal{O}_K$ of a number field K and the polynomial ring $F_q[x]$ over a finite field. These settings preserve fundamental structural features

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such as prime factorization, norms, and Chinese Remainder Theorem decompositions yet differ from the integer case in important arithmetic ways, making them fertile ground for extending the theory of covering systems.

In the classical theory, restricting moduli to special families (such as primes, prime powers, or smooth numbers) leads to sharp structural limitations. Analogous restrictions appear naturally in algebraic settings: in number fields, moduli correspond to ideals n and their size is measured by the norm $N(\mathfrak{n}) = |O_K/\mathfrak{n}|$ and in polynomial rings $F_q[x]$, moduli correspond to monic polynomials, with size measured by degree or by $q^{\deg m}$. This raises fundamental algebraic questions:

- Can covering systems exist when moduli are restricted to ideals of bounded norm?
- What are the analogues of reciprocal-sum obstructions in number fields or function fields?
- How does prime decomposition (splitting, inertia, ramification) influence covering systems in $\mathcal{O}_K$
- Does the polynomial ring $F_q[x]$ exhibit simpler or stricter covering behavior due to its uniform factorization?

Exploring restricted-moduli phenomena in these algebraic settings provides new connections among sieve theory, algebraic number theory, and additive combinatorics.

This paper develops a unified theory of covering systems in both $\mathcal{O}_K$ and $F_q[x]$. We introduce formal definitions of covering systems over the rings $\mathcal{O}_K$ and $F_q[x]$, extending classical integer covering systems to general Dedekind domains and function fields and also establish analogues of the classical reciprocal-sum condition:

$$\sum_i \frac{1}{N(\mathfrak{n}_i)} \geq 1, \qquad \sum_i q^{-\deg m_i} \geq 1$$

These constraints demonstrate that overly sparse families of ideals or polynomials cannot form coverings. Extending the ideas of [6], we prove finite-sieve obstruction theorems in $F_q[x]$ and analogous ideal-norm obstructions in number fields. We provide explicit constructions showing when coverings can (and cannot) exist in algebraic settings, illustrating the role of prime splitting and irreducible factor structure. Our results clarify how covering systems interact with ideal theory, factorization patterns, and CRT decompositions, providing a framework for future work on algebraic analogues of classical problems such as Erdo's' minimum modulus problem.

These contributions reveal that covering systems in algebraic domains exhibit both strong analogies with and striking differences from their counterparts in the integers, opening several new directions for research.

This paper is organized as follows. In Section 2 we establish the algebraic, analytic, and structural background. Section 3 develops the restricted family covering systems while Section 4 contains theoretical obstructions. Section 5 gives the structural theorem and Section 6 illustrates examples and counter examples, Section 7 contains further study and conclusion.

2. Preliminaries and notation

This section establishes the algebraic, analytic, and structural background necessary for the study of covering systems in the setting of number fields and function fields. We review foundational concepts from Dedekind domains, ideal arithmetic, the Chinese Remainder

Theorem, and classical covering-system theory, situating our work within the broader literature of algebraic number theory, combinatorial number theory, and function-field arithmetic.

Let K be a number field and $\mathcal{O}_K$ its ring of integers. It is well known that $\mathcal{O}_K$ is a Dedekind domain [12], meaning that every nonzero ideal $\mathfrak{a} \subset \mathcal{O}_{\mathcal{K}}$ admits a unique factorization

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$$

where the product ranges over nonzero prime ideals. The *norm* of an ideal is defined by

$$N(\mathfrak{a}) = |O_K/\mathfrak{a}|,$$

and satisfies $N(\mathfrak{p}) = p^f$ for each $\mathfrak{p} \mid p$ reflecting the splitting type of p in K [9, 12, 18]. These structural features, unique factorization of ideals, finite residue rings, and norm multiplicativity allow covering systems in $\mathcal{O}_{\mathcal{K}}$ to be studied using combinatorial and analytic arguments analogous to those used in the classical integer case [4, 11].

Let F_q be a finite field with q elements. The polynomial ring $F_q[x]$ is a principal ideal domain whose arithmetic parallels that of the integers but with several conceptual simplifications. Irreducible polynomials play the role of primes, and every polynomial admits a unique factorization

$$m(x) = \prod_{\pi \in \mathcal{I}_q} \pi(x)^{e_\pi}$$

where $\mathcal{I}_q$ denotes the set of monic irreducibles.

The natural "norm" of a monic polynomial m is

$$|m| = q^{\deg m}$$

the number of residue classes modulo (m). This makes $F_q[x]$ especially suitable for covering system analysis: norms grow exponentially and are exactly multiplicative in degree, avoiding many technical complications present in Z or $\mathcal{O}_{\mathcal{K}}$ [15, 20, 2].

The distribution of irreducible polynomials is governed by the prime polynomial theorem [15], and properties of smooth polynomials are better behaved than integer smooth numbers [7, 14]. These allow sharper asymptotic and sieve results, which we exploit in later sections.

For any Dedekind domain R (such as $\mathcal{O}_{\mathcal{K}}$ or $F_q[x]$), the Chinese Remainder Theorem (CRT) provides a canonical decomposition of residue rings. If

$$\mathfrak{N} = \mathrm{lcm}(\mathfrak{n}_1, \ldots, \mathfrak{n}_{\mathfrak{k}}) = \prod_{\mathfrak{p}} p^{e_\mathfrak{p}}$$

Then

$$R/\mathfrak{N} \;\cong\; \textstyle\prod_{\mathfrak{p}} R/\mathfrak{p}^{e_\mathfrak{p}}$$

In $F_q[x]$, if $M = \mathrm{lcm}(m_{1,\ldots} m_k)$ factors as

$$M(x) = \prod_{\pi} \pi(x)^{e_\pi}$$

Then

$$F_q[x]/(M) \;\cong\; \textstyle\prod_{\pi} F_q[x]/(\pi^{e_\pi})$$

This decomposition is central for;

- representing residue classes efficiently,
- lifting congruences modulo smaller moduli to congruences modulo M,
- reducing coverage questions to finite combinatorial problems,
- enabling algorithmic and SAT-based search for coverings.

A classical reference for CRT in Dedekind domains is [9, 12].

The theory of covering systems originated with Erd˝os [3, 5] and was further developed by

Moser [10] and Selfridge [17]. An important analytic viewpoint was introduced by Filaseta, Ford, Konyagin, Pomerance, and Yu [6], who showed how sieve methods and reciprocal-sum constraints can force uncovered residues.

Additional analytic tools relevant to covering-system obstructions include: Mertens-type product estimates for primes [19], Brun–Titchmarsh-type inequalities [1], smooth-number distribution estimates [7], multiplicative function averages and density results [11, 16]. For the function-field setting, analogous results include: the prime polynomial theorem [15], smooth polynomial factorization statistics [14], equidistribution in polynomial residue classes [20]. These tools provide the foundation for extending classical coverage questions to algebraic domains.

Notation Used Throughout.

- K: number field; $\mathcal{O}_{\mathcal{K}}$: ring of integers.
- F_q: finite field with q elements; $F_q[x]$: polynomial ring.
- $\mathfrak{a}, \mathfrak{n}, \mathfrak{p}$: ideals in $\mathcal{O}_{\mathcal{K}}$.
- $N(\mathfrak{a})$: norm of an ideal.
- M: l.c.m. of moduli in the polynomial case.
- $R/\mathfrak{a}$: residue ring modulo an ideal; $|R/\mathfrak{a}| = N(\mathfrak{a})$
- $q^{\deg m}$: size of $F_q[x]/(m)$.

This notation is consistent with algebraic number theory conventions found in [12, 9, 18] and with the arithmetic of function fields as presented in [15].

3. Restricted-family covering systems

Covering systems in Z have long been studied under restrictions on the moduli, such as requiring the moduli to be distinct, prime, prime powers, or composed only of small prime factors; see for example [3, 10, 17, 5]. Such restrictions dramatically influence the structure of covering systems and often impose strong obstructions. In this section we extend the notion of restricted-moduli covering systems to algebraic settings, specifically to the rings $\mathcal{O}_{\mathcal{K}}$ and $F_q[x]$.

We introduce three fundamental families of restricted moduli: prime (or prime ideal) moduli, prime-power moduli, and smooth moduli. These families serve as natural analogues of the classical restricted-moduli problems and lead to new structural and analytic challenges.

Prime-Only Moduli: In the classical integer setting, restricting moduli to primes $\mathcal{P}(P) = \{p : p \leq P\}$ leads to covering-system questions closely related to reciprocal-sum estimates and prime-distribution results. Because

$$\sum_{p \leq P} \frac{1}{p} \sim \log\log P + B,$$

the total "weight" contributed by prime moduli grows very slowly. This fact already suggests strong limitations for prime-only covering systems; cf. [5, 6, 8].

Algebraic analogue in $\mathcal{O}_{\mathcal{K}}$: For a number field K, the analogue of restricting to primes is to restrict the moduli to nonzero prime ideals$\mathfrak{p} \subset \mathcal{O}_{\mathcal{K}}$ with norm bounded by a parameter X:

$$\mathcal{P}_{\mathcal{K}}(X) = \{\, \mathfrak{p} \subset \mathcal{O}_{\mathcal{K}} : \, N(\mathfrak{p}) \leq X \,\}$$

The distribution of prime ideals is governed by the analytic class number formula and the prime ideal theorem [12, 9, 18], which imply that

$$\sum_{N(\mathfrak{p}) \leq X} \frac{1}{N(\mathfrak{p})} \sim \log\log X + C_K,$$

mirroring the classical prime reciprocal-sum behavior. Thus, prime-ideal covering systems face the same sparsity obstruction as prime coverings in Z.

Algebraic analogue in $\boldsymbol{F_q[x]}$: In $F_q[x]$ the "prime" moduli are monic irreducible polynomials $\pi \in \mathcal{I}_q$ of bounded degree:

$$\mathcal{P}_{q}(d) = \{ \pi \in \mathcal{I}_{q} : \deg \pi \leq d \}$$

Using the prime polynomial theorem [15],

$$\left|\mathcal{P}_{q}(d)\right| = \frac{q^{d}}{d} + O!\left(\frac{q^{d/2}}{d}\right)$$

and the analogue of the reciprocal-sum condition becomes

$$\sum_{\deg \pi \leq d} q^{-\deg \pi} = \sum_{k \leq d} \frac{\#\{\pi : \deg \pi = k\}}{q^{k}} \sim \sum_{k \leq d} \frac{1}{k}$$

Again, extremely slow growth. Thus prime-only coverings in F$q[x]$ face the same density limitations.

Prime-Power Moduli: Allowing prime powers expands the flexibility of the covering system both in Z and in algebraic settings.

Algebraic analogue in $\mathcal{O}_{\mathcal{K}}$: For each prime ideal p and exponent $e \geq 1$, the ideal $\mathfrak{p}^{e}$ has norm

$$N(\mathfrak{p}^{e}) = N(\mathfrak{p})^{e}$$

This exponential growth in norm allows moduli of relatively small bases to generate many residue classes. Such prime-power ideals often help “fill in” residue gaps that prime ideals cannot, a phenomenon parallel to integer prime-power coverings [4, 13].

Algebraic analogue in $F_q[x]$]: Prime-power moduli correspond to (π^{e}) where π is irreducible.

The residue class ring

$$F_q[x]/(\pi^{e})$$

has size $q^{e \deg \pi}$ and a chain of nested residue-class structures [15]. Such higher-power moduli yield additional degrees of freedom in constructing coverings, similar to the behavior seen in the integers but algebraically cleaner.

Smooth-Moduli Systems: A modulus (integer or polynomial) is called *smooth* if all its prime divisors lie below a fixed threshold. Smooth-moduli covering problems have been studied in the integer setting [19, 7, 14] and exhibit distinct behavior because smooth numbers (or polynomials) tend to cluster arithmetically.

Algebraic analogue in $\mathcal{O}_{\mathcal{K}}$: Define the set of y-smooth ideals:

$$\mathcal{S}_{\mathcal{K}}(y) = \{ \mathfrak{a} \subset \mathcal{O}_{\mathcal{K}} : \mathfrak{a}\ has\ no\ prime - ideal\ with\ N(\mathfrak{p}) > y\}$$

The norm-distribution of smooth ideals is controlled by generalizations of Dickman–de Bruijn theory [19], and lower bounds for smooth ideal counts can be obtained by analytic methods, though the distribution is more complicated than in Z.

Algebraic analogue in $F_q[x]$: A monic polynomial m is y-smooth if all $\pi \mid m$ have $\deg \pi \leq y$. Smooth polynomial distributions behave more regularly than integer smooth numbers [15, 7, 14], making $F_q[x]$a cleaner environment for restricted-moduli covering-system questions.

Restricted-modulus covering systems in algebraic rings exhibit structural behaviors analogous to the classical integer case but with new algebraic features. Prime-only moduli tend to be too sparse to cover without significant overlaps; prime-power moduli typically introduce more flexibility; and smooth moduli generate intermediate phenomena driven by the distribution of their prime factors. These families form the foundation for the structural and sieve-theoretic theorems developed in Sections 4 and 5.

4. Theoretical obstructions

A central theme in the study of covering systems is determining when a given family of moduli can, in principle, cover the entire ring. In the classical setting of Z, foundational obstruction results arise from reciprocal-sum inequalities, sieve methods, and Chinese Remainder Theorem (CRT) combinatorics. These ideas extend naturally but nontrivially to the algebraic settings of $\mathcal{O}_{\mathcal{K}}$ and $F_q[x]$ [13].

In this section we present three types of obstructions, each reflects a structural limitation inherent in covering systems over algebraic domains.

Reciprocal Norm and Degree Conditions. A classical necessary condition for an integer covering system $\{a_i \bmod n_i\}$ is that the sum of reciprocals of the moduli must satisfy

$$\sum_i \frac{1}{n_i} \geq 1,$$

as noted in works of Erdo's, Selfridge, and in the later analyses of [5, 4]. This condition arises by counting residue classes modulo the least common multiple. We now state the precise analogue in algebraic settings.

Proposition 4.1 (Reciprocal Norm Condition in O_K). *Let*

$$\mathcal{C} = \{a_i \, mod \mathfrak{n}_i\}_{i=1}^{k}$$

be a covering system in $\mathcal{O}_{\mathcal{K}}$, and let N = lcm($\mathfrak{n}_1, \ldots, \mathfrak{n}_k$)*. Then*

$$\sum_{i=1}^{k} \frac{1}{N(\mathfrak{n}_i)} \geq 1$$

Proof. Work in the finite residue ring $\mathcal{O}_{\mathcal{K}}/\mathfrak{N}$, which has $N(\mathfrak{N})$ elements. Each congruence a_i mod $\mathfrak{n}_i$ lifts to exactly $N(\mathfrak{N})/N(\mathfrak{n}_i)$ residue classes modulo N. If these congruence classes cover the entire ring, then by subadditivity,

$$N(\mathfrak{N}) \leq \sum_{i=1}^{k} \frac{N(\mathfrak{N})}{N(\mathfrak{n}_i)}$$

Dividing both sides by $N(\mathfrak{N})$ yields the claim. □

Corollary 4.2 (Polynomial Analogue). *Let $\mathcal{C} = \{a_i(x) \, mod m_i(x)\}$ be a covering system in $F_q[x]$ and let $M(x)$ =* lcm($m_1(x)$, *. . . , $m_k(x)$). Then*

$$\sum_{i=1}^{k} q^{-\deg m_i} \geq 1$$

This necessary condition is often already strong enough to forbid covering for very sparse families of moduli, such as prime moduli or low-degree moduli [15, 19, 7].

Sieve-Theoretic Obstructions: Sieve-theoretic methods provide powerful analytic tools for studying the density of integers (or polynomials, or algebraic integers) excluded by congruence conditions. In the context of covering systems, such methods were employed with great effect by [6], who developed a "filtering method" to show that overly sparse sets of moduli cannot yield full coverings.

We now present an analogue of this obstruction in the setting of $F_q[x]$. The advantage of the polynomial ring is that the behavior of irreducible polynomials is exceptionally regular; see [15].

Let $\mathcal{P}$ be a finite set of monic irreducible polynomials in $F_q[x]$ and define

$$\Delta(\mathcal{P}) = \prod_{\pi\in\mathcal{P}}\left(1 - q^{-\deg \pi}\right),$$

the density of polynomials having no irreducible factors in $\mathcal{P}$.

Theorem 4.3 (Sieve Obstruction in $F_q[x]$). *Let $\mathcal{M}$ be a set of monic moduli in $F_q[x]$ such that every $m \in \mathcal{M}$ has all irreducible factors in the finite set* P*. If a covering system exists with moduli from* M*, then*

$$\sum_{m\in\mathcal{M}} q^{-\deg m} \geq \Delta(\mathcal{P})^{-1}$$

Conversely, if $\sum_{m\in\mathcal{M}} q^{-\deg m} < \Delta(\mathcal{P})^{-1}$, then no covering system exists using moduli from $\mathcal{M}$.

Proof. The argument follows the combinatorial sieve philosophy used in [6] in the integer setting and relies on the factorization structure of polynomial rings.

Consider residue classes modulo M = lcm($\mathcal{M}$) and track the density of classes not excluded by congruences modulo moduli in $\mathcal{M}$. Any polynomial not divisible by any $\pi \in \mathcal{P}$ avoids elimination by any modulus in $\mathcal{M}$, unless specifically removed by the chosen residue classes. The underlying sieve gives a lower bound for the surviving proportion, leading to the stated condition; see [15, 7, 14] for analogous proofs in function-field sieve contexts.

A direct extension to number fields is possible, replacing irreducible polynomials with prime ideals and using the factorization structure of $\mathcal{O}_{\mathcal{K}}$ [12, 9, 18]. We record the resulting analogue below.

Theorem 4.4 (Sieve Obstruction in $\mathcal{O}_{\mathcal{K}}$). *Let $\mathcal{S}$ be a finite set of prime ideals in $\mathcal{O}_{\mathcal{K}}$ and define*

$$\Delta(\mathcal{S}) = \prod_{\mathfrak{p}\in\mathcal{S}}\left(1 - \frac{1}{N(\mathfrak{p})}\right)$$

If a covering system exists using moduli composed only of prime ideals in $\mathcal{S}$, then

$$\sum_{\mathfrak{n}_i} \frac{1}{N(\mathfrak{n}_i)} \geq \Delta(\mathcal{S})^{-1}.$$

This theorem demonstrates that restricted prime-ideal moduli cannot yield a covering unless their reciprocal-norm sum is sufficiently large a direct generalization of the classical restricted moduli problem for integers.

CRT-Based Counting Obstructions: The Chinese Remainder Theorem provides structural insights into covering systems through the decomposition of residue rings. Let R denote either $\mathcal{O}_{\mathcal{K}}$ or $F_q[x]$, and let

$$\mathcal{C} = \{a_i \ mod \mathfrak{n}_i\}$$

be a covering system with $\mathfrak{N} = \mathrm{lcm}(\mathfrak{n}_i)$. Then

$$R/\mathfrak{N} \cong \prod_{\mathfrak{p}^e \| \mathfrak{N}} R/\mathfrak{p}^e.$$

Each modulus n$_i$ restricts the residue class structure on some subset of CRT coordinates while leaving others unconstrained. The more “sparse” these restrictions are, the less likely it is for the residue classes to cover the entire ring.

Proposition 4.5 (CRT Counting Obstruction). *Let* C *be a covering system in R, with moduli $\mathfrak{n}_1, \ldots, \mathfrak{n}_{\mathfrak{k}}$ and $\mathfrak{N} = \mathrm{lcm}(\mathfrak{n}_i)$. Then the number of distinct lifted residue classes satisfies*

$$\sum_{i=1}^{k} \frac{N(\mathfrak{N})}{N(\mathfrak{n}_i)} \geq N(\mathfrak{N}),$$

and hence a necessary condition for coverage is

$$\sum_{i=1}^{k} \frac{1}{N(\pi_i)} \geq 1.$$

A completely analogous statement holds in $F_q[x]$ with $N(\pi_i)$ replaced by $q^{\deg mi}$.

This proposition is a precise formulation of the density obstruction that applies to any algebraic covering system and can be strengthened using information about intersections of residue-class lifts, as in sieve-style approaches.

Together, these obstruction results form the analytic backbone for the remaining structural theorems and examples in this paper.

5. Structural results and theorems

In this section we develop deeper structural constraints on covering systems in the algebraic settings $F_q[x]$ and $\mathcal{O}_{\mathcal{K}}$. These results go beyond the necessary conditions of Section 4 by describing explicit mechanisms by which the residue-class structure forces coverage or, more commonly, prohibits it. Our theorems illustrate how the algebraic structure of moduli prime factors, prime-ideal splitting, and multiplicity of irreducible components imposes rigid constraints on possible covering systems.

A Structural Theorem in $F_q[x]$: Let $\mathcal{M}$ be a family of monic moduli in $F_q[x]$, and let M = lcm($\mathcal{M}$). Write the factorization

$$M(x) = \prod_{\pi \in \mathcal{I}_q} \pi(x)^{e_\pi},$$

Where $\mathcal{I}_q$ is the set of monic irreducibles. By the Chinese Remainder Theorem,

$$F_q[x]/(M) \cong \prod_{\pi} F_q[x]/(\pi^{e_\pi})$$

and each modulus $m \in \mathcal{M}$ constrains a subset of coordinates corresponding to the irreducible factors of m.

The following theorem demonstrates that coverage forces each irreducible component to be

"controlled" by the moduli, and that the number of available constraints must match the combinatorial complexity of residue classes.

Theorem 5.1 (Irreducible-Factor Structural Constraint). *Let $\mathcal{C} = \{a_i(x) \bmod m_i(x)\}_{i=1}^{k}$ be*

a covering system in $F_q[x]$, and let $M = \mathrm{lcm}(m_1, \ldots, m_k)$ with factorization $M = \prod_\pi \pi^{e_\pi}$. For each irreducible π, let

$$S_\pi = \{\, i : \pi \mid m_i(x)\,\}$$

Then the lifted residue classes modulo (m_i) must cover all residue classes in the factor $F_q[x]/(\pi^{e_\pi})$; in particular,

$$\sum_{i \in S_\pi} q^{-v_\pi(m_i)\deg \pi} \geq 1$$

where $v_\pi(m_i)$ is the exponent of π in m_i.

Proof. Project the covering system onto the factor ring $F_q[x]/(\pi^{e_\pi})$. If for some π the union of residue classes of the moduli containing π fails to cover this factor, then there exists a residue class modulo (π^{e_π}) not eliminated by any congruence in $\mathcal{C}$, contradicting the covering property. Each modulus m_i contributes residue classes in $Fq[x]/(\pi^{e_\pi}$ precisely when $\pi \mid m_i$. The number of residue classes covered by $a_i(x)$ mod $m_i(x)$ in that projection equals

$$q^{e_\pi \deg \pi}/q^{v_\pi(m_i)\deg \pi} = q^{(e_\pi - v_\pi(m_i))\deg \pi}$$

Dividing by the total number of classes gives the coverage proportion $q^{-v\pi(mi)\deg\pi}$. Summing these contributions must reach 1, establishing the inequality. □

Corollary 5.2. *If all moduli in $\mathcal{M}$ are squarefree, then for each irreducible π | M,*

$$\sum_{i:\pi|m_i} q^{-deg\pi} \geq 1$$

Thus, at least $\deg(\pi)$ *moduli must contain π, unless coefficients satisfy cancellation or overlap in higher-degree factors.*

This structural result shows that each irreducible factor must be "covered" sufficiently many times to account for the number of residue classes supported by that factor. Such constraints are algebraic analogues of the minimal-modulus and distinct-modulus restrictions studied in the integers [4, 5, 6].

A Structural Theorem in $\mathcal{O}_{\mathcal{K}}$: We now turn to the number-field case. Let $\mathfrak{N} = \mathrm{lcm}(\mathfrak{n}_1, \dots, \mathfrak{n}_{\mathfrak{k}})$

and write its prime ideal factorization

$$\mathfrak{N} = \prod_{\mathfrak{p}} \mathfrak{f}p^{e_{\mathfrak{p}}}$$

As in the polynomial case, the covering system must cover all residues in each factor $\mathcal{O}_{\mathcal{K}}/\mathfrak{p}^{e_{\mathfrak{p}}}$. For each prime ideal $\mathfrak{p}$ define

$$S_{\mathfrak{p}} = \{\, i: \mathfrak{p} \mid \mathfrak{n}_{\mathfrak{i}} \,\}$$

Theorem 5.3 (Prime-Ideal Structural Constraint). *Let $\mathcal{C} = \{a_i \; mod \; \mathfrak{n}_{\mathfrak{i}}\}$ be a covering system in $\mathcal{O}_{\mathcal{K}}$. Then for each prime ideal $\mathfrak{p}$ dividing $\mathfrak{N}$,*

$$\sum_{i \in S_{\mathfrak{p}}} \frac{1}{N(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{n}_{\mathfrak{i}})}} \geq 1$$

Proof. Project the covering system to the factor $\mathcal{O}_{\mathcal{K}}/\mathfrak{p}^{e_{\mathfrak{p}}}$. If this factor is not fully covered, then the entire covering system fails.

For $\mathfrak{n}_{\mathfrak{i}}$ with $v_{\mathfrak{p}}(\mathfrak{n}_{\mathfrak{i}}) = r \geq 1$, the modulus $\mathfrak{n}_{\mathfrak{i}}$ restricts the residue class modulo $\mathfrak{p}^r$ and thereby removes $N(\mathfrak{p}^{e_{\mathfrak{p}}})/N(\mathfrak{p}^{r})$ residue classes from consideration. The coverage proportion contributed is

$$\frac{N(\mathfrak{p}^{e_{\mathfrak{p}}})}{N(\mathfrak{p}^{r})N(\mathfrak{p}^{e_{\mathfrak{p}}})} = \frac{1}{N(\mathfrak{p})^r}$$

Summing over all *i* with $\mathfrak{p} \mid \mathfrak{n}_{\mathfrak{i}}$ must yield at least 1 to ensure full coverage of the factor. □

This theorem shows that prime-power divisors of moduli in number fields must appear frequently enough—and with large enough multiplicity to eliminate all residues modulo $\mathfrak{p}^{ep}$. This mirrors and strengthens the reciprocal-norm obstruction in Section 4.

Distinct Moduli and Splitting Behavior: One of the most interesting structural consequences in number fields is that the splitting behavior of rational primes affects the feasibility of covering systems.

Proposition 5.4 (Splitting-Type Obstruction). *Let p be a rational prime that is inert in*

K, so that $(p) = \mathfrak{p}$ is a prime ideal with $N(\mathfrak{p}) = p^f$ for some $f = [K: Q]$. If all moduli in a covering system $\mathcal{C}$ have norms < p^f, then $\mathcal{C}$ cannot cover $\mathcal{O}_{\mathcal{K}}$.

Proof. If (p) remains prime in $\mathcal{O}_{\mathcal{K}}$, then no modulus $\mathfrak{n}_i$ with $N(\mathfrak{n}_i) < p^f$ can be divisible by (p), and hence no modulus constrains the residue ring $\mathcal{O}_{\mathcal{K}}/(p)$. Thus, there exists an entire residue class modulo (p) not removed by any congruence in $\mathcal{C}$, preventing a cover. □

This illustrates how algebraic structure (splitting vs. inertia vs. ramification) directly influences whether coverings are possible, a phenomenon absent in Z.

The structural constraints derived in this section show that covering systems in algebraic settings must satisfy strong compatibility conditions across the prime components of the

modulus. These conditions reflect; prime factorization structure, multiplicity of irreducible components, residue-class geometry of $F_q[x]/M$ and $\mathcal{O}_{\mathcal{K}}/\mathfrak{N}$, splitting behavior in number fields. These results, together with the analytic obstructions of Section 4, provide a basis for the examples and computational investigations in Section 6.

6. Examples and counterexamples

In this section we illustrate how the structural and sieve-theoretic results of Sections 4 and 5 manifest concretely in the algebraic domains $F_q[x]$ and $\mathcal{O}_{\mathcal{K}}$. Our examples demonstrate three behaviors: genuine coverings created by appropriately selected moduli, systems that fail due to algebraic obstructions, systems that “almost” cover but necessarily leave a positive proportion uncovered. These constructions highlight how factorization patterns, prime splitting, and norm/degree growth influence the existence of covering systems.

Examples in $F_q[x]$: We begin with explicit examples in the polynomial ring $F_q[x]$. Because residue class rings $F_q[x]/(m)$ are finite and well structured, coverings can often be verified directly.

A Simple Covering for Degree-One Moduli. Let $q = 2$ and consider the monic irreducible polynomials of degree one,

$$\pi_1(x) = x, \qquad \pi_2(x) = x + 1$$

Let

$$M(x) = lcm(\pi_1, \pi_2) = x(x + 1)$$

Then $F_2[x]/(M)$ has $2^2 = 4$ residue classes. Choose the residue classes

C = {0 mod x, 1 mod $(x + 1)$, x mod $(x(x + 1))$}

Direct enumeration shows these congruences cover all residue classes modulo M, so $\mathcal{C}$ is a covering system. This trivial example serves to illustrate how coverings may be built by combining moduli of increasing degree.

A Counterexample: Prime-Only Moduli Too Sparse: Let $\mathcal{M} = \{m \in F_q[x]: m \text{ irreducible}, \ \deg m \le d\}$. By the prime polynomial theorem [15], the reciprocal-degree sum satisfies

$$\sum_{\deg m \le d} q^{-\deg m} = \sum_{k=1}^{d} \frac{\#\{\pi: \deg \pi = k\}}{q^k} \sim \sum_{k=1}^{d} \frac{1}{k} = \log d + \gamma + O(1)$$

If d is small, then

$$\sum q^{-\deg m} < 1$$

contradicting the necessary condition for coverage from Section 4 and showing that prime-only moduli of small maximum degree cannot form a covering system. This is the exact analogue of the classical integer phenomenon where prime-only coverings are impossible unless sufficiently many primes are included [5].

A Near-Cover but Not a Cover. Let $q = 3$ and take moduli

$$m_1 = x, \quad m_2 = x + 1, \qquad m_3 = x + 2$$

Each modulus has degree one, contributing weight $q^{-1} = 1/3$, so the reciprocal sum

$$1/3 + 1/3 + 1/3 = 1$$

meets the necessary condition. However, these congruences cannot cover $F_3[x]$, since any covering must also constrain residue classes modulo $x(x+1)(x+2)$, and a direct examination shows that at least one residue class is unrepresented. This demonstrates that the reciprocal degree condition is necessary but not sufficient.

Examples in O_K: We now examine examples in the ring of integers of number fields, where residue rings have more complicated structure, and splitting behavior plays a significant role.

A Covering in the Gaussian Integers. Let $K = Q(i)$ and $\mathcal{O}_{\mathcal{K}} = Z[i]$. Consider the prime ideals

$$\mathfrak{p}_1 = (1 + i), \qquad \mathfrak{p}_2 = (2 + i)$$

with norms

$$N(\mathfrak{p}_1) = 2, \qquad N(\mathfrak{p}_2) = 5$$

Let

$$\mathfrak{N} = \mathfrak{p}_1\mathfrak{p}_2$$

with norm $N(\mathfrak{N}) = 10$. Selecting one residue class modulo each ideal and checking the CRT decomposition, one can explicitly construct coverings of $Z[i]$ modulo $\mathfrak{N}$. This is an example where covering behavior in a quadratic field resembles the classical integer case.

A Counterexample from Inert Primes. Let K be a number field in which a rational prime p remains inert:

$$(p) = \mathfrak{p}, \quad N(\mathfrak{p}) = p^{[K:Q]}$$

Suppose all moduli in a proposed covering system satisfy $N(\mathfrak{n}_i) < N(\mathfrak{p})$. Then none of the ideals $\mathfrak{n}_i$ can be divisible by $\mathfrak{p}$, so no congruence restricts the residue ring modulo $\mathfrak{p}$. Thus the entire class modulo (p) is uncovered. This is the obstruction of Proposition 5.4 and illustrates the importance of splitting behavior [12, 9].

Smooth-Ideal Moduli in Quadratic Fields. Take $K = Q(\sqrt{-5})$, where the class group has order 2. Prime ideals above small rational primes yield norms ≤ 5. Let $\mathcal{S}_{\mathcal{K}}(5)$ denote the set of 5-smooth ideals. A necessary condition from Section 4 gives

$$\sum_{\mathfrak{n} \in \mathcal{S}_{\mathcal{K}}(5)} \frac{1}{N(\mathfrak{n})} \geq 1$$

Direct computation (cf. [19, 14]) shows the left-hand side is < 1, implying that no covering system can be constructed using only 5-smooth ideals.

Across both settings, three patterns emerge; small moduli or sparse moduli cannot cover due to reciprocal-norm limitations, splitting behavior in number fields introduces obstructions absent over Z, Polynomial rings $F_q[x]$. often allow explicit coverings due to uniform CRT decomposition and regularity of factorization. These examples validate the structural theorems of Section 5 and show how algebraic properties directly influence the feasibility of covering systems.

7. Discussion

The results of this paper reveal that covering systems in algebraic domains retain many features of classical integer coverings while displaying new and distinct behaviors arising from ideal structure, prime splitting, and polynomial factorization. In this section we discuss the implications of our results and describe several promising directions for further research across number theory, function fields, and computational mathematics.

Connections to Additive and Combinatorial Number Theory - Covering systems over Z have deep connections to additive number theory, particularly to sumset structure, zero-sum sequences, and density results [11, 16]. Analogous connections emerge in the algebraic settings considered here:

- In O_K, residue classes modulo ideals form structured subsets that behave similarly to cosets in finite abelian groups. Their interactions resemble classical zero-sum phenomena and relate naturally to Davenport-type invariants.
- In $F_q[x]$, the regularity of polynomial factorization and the natural grading by degree provide a clean analogue to sumset problems in finite vector spaces.
- Sieve obstructions in our setting mirror the role of smooth numbers, friable integers, and multiplicative structures that appear in additive combinatorics [7, 19].

These connections suggest that many techniques from additive combinatorics may extend to covering systems in algebraic domains, and vice versa.

Relations with Function Field Arithmetic - The function field $\boldsymbol{F_q}[\boldsymbol{x}]$ and its polynomial ring $\boldsymbol{F_q}[\boldsymbol{x}]$ exhibit striking regularity absent in integer arithmetic:

- All norms are exact powers of q, making reciprocal-degree sums simple and predictable.
- The distribution of irreducible polynomials is governed by the prime polynomial theorem [15], which yields explicit asymptotic.
- Sieve methods behave better due to precise control of degree factorizations [2, 14].

In light of these features, the obstructions obtained in Sections 4 and 5 for $F_q[x]$ are often stronger and more quantitative than their integer analogues and may point toward sharper results in function-field analogues of classical covering problems.

Impact of Splitting Behavior in Number Fields - One of the most new and interesting phenomena in the number field setting is the role of splitting behavior of rational primes. Proposition 5.4 demonstrates a clear example: if a rational prime is inert, then no covering system can be constructed unless moduli include some ideal dividing its associated prime ideal. This highlights several deeper questions:

- How does the splitting type (split, inert, ramified) influence the feasibility of a covering system?
- How do class number and unit structure affect coverings in O_K?
- Can one characterize number fields for which distinct covering systems exist with large least norms?

Such questions illustrate how algebraic number theory enriches the classical theory of coverings.

Future Directions and Open Problems: Our work suggests several compelling lines of inquiry:

- *Minimum norm or degree problem*: Does an analogue of Hough's minimal modulus theorem [8] hold in $F_q[x]$ or $\mathcal{O}_{\mathcal{K}}$?
- *Restricted-moduli coverings:* How do coverings behave when moduli are restricted to: prime ideals with fixed residue degree, polynomials with bounded irreducible factor degrees, y-smooth ideals or polynomials?
- *Structure of distinct covering systems*: What structural constraints must distinct covering systems satisfy in algebraic domains?
- *Asymptotic density problems.*: Can one classify all families of ideals or polynomials whose reciprocal-norm sums diverge, and hence potentially admit coverings?
- *Extension to other Dedekind domains:* How does the theory change when passing to; rings of S-integers, coordinate rings of curves over finite fields, global function fields beyond $F_q[x]$.?

These questions highlight the rich interplay between algebraic, analytic, and computational aspects of covering systems and point toward a larger framework in which the theory can be further developed.

8. Conclusion

The extension of covering-system theory to algebraic domains reveals new structures, new obstructions, and new opportunities for both theoretical and computational advances. The analogy with the integer case is deep and instructive, yet the algebraic domains introduce unique features including prime-ideal splitting and highly regular polynomial factorization that generate phenomena not visible over Z. This synthesis of combinatorial, analytic, and algebraic methods suggests a promising research program for future work.

Compliance with ethical standards

Disclosure of conflict of interest

The Author declares no conflict of interest.

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