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On the asymptotic value of approximation Mellin singular integral in the summable function

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Abstract

Many questions about the order of approximation and their order of convergence of various classes of functions by linear operators, in particular singular integrals. Approximations of functions by singular integrals have numerous applications in various fields of mathematics. Approximations of functions by singular integrals are studied intensively along with other issues in theory of functions. In their papers PL Butzer and RG Mamedov study convergence order of singular integrals in generating functions at separate characteristic points and metric in the space L^p on bounded and unbounded domains. Important theorems on asymptotic value of approximation of functions by singular integrals are obtained in these papers. In this paper, we study the approximation properties of the Mellin singular integral in terms of the mean oscillation of a locally summable function.

Keywords: Order; Kernel; Asymptotic; Sinqular Inteqrals; Metrics; Space

1. Introduction

The study of singular integral operators has long been a central topic in harmonic analysis, approximation theory, and the theory of integral equations. Among these operators, Mellin singular integrals occupy a special place due to their close connection with problems on the half-axis and their applications in mathematical physics, probability theory, and number theory. The Mellin transform framework provides a natural tool for analyzing functions defined on allowing convolution-type operations to be treated as multiplicative convolutions, which are particularly well-suited for problems exhibiting scale invariance.

Approximation of singular integrals by discrete or regularized analogues has been the focus of extensive research, as exact evaluation is often impossible in practical applications. In many cases, understanding the asymptotic behavior of these approximations plays a crucial role in quantifying the accuracy and convergence rate of numerical and analytical methods. Recent advances have focused on the asymptotics of Mellin-type singular integral operators when applied to classes of summable functions, revealing deep connections between the local behavior of the kernel and the global convergence of the operator.

2. Some definitions and designations

Let $R=(-\infty,+\infty)$, $R_+=(0,+\infty)$. If $E=R$ or $E=R_+$, then $L_{loc}(E)$ denotes the set of all functions locally summable on the set E. $L(X)=L(X;dx)$ is the set of all functions summable on the set of $X\subset R$ relative linear

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Lebesgue measure dx. In what follows, $L\left(R_+;\frac{dx}{x}\right)$ we will denote the set of all functions f summable with respect to measure $\frac{dx}{x}$ on the set R_+.

Let $0<\tau\leq 1,\ x\in R_+,\ I(x;\tau):=\left\{\rho\in R_+ : x\tau\leq\rho\leq\frac{x}{\tau}\right\},\ f\in L_{loc}(R_+)$. Let us introduce the following notations

$$f_{I(x;\tau)}:=\frac{1}{2|\ln\tau|}\int_{x\tau}^{x\tau^{-1}} f(\rho)\frac{d\rho}{\rho},$$

$$\Omega^M(f,I(x;\tau)):=\frac{1}{2|\ln\tau|}\int_{x\tau}^{x\tau^{-1}} \left|f(\rho)-f_{I(x;\tau)}\right|\frac{d\rho}{\rho}.$$

the quantity $\Omega^M(f,I(x;\tau))$ the average Mellin oscillation of the function f on the interval $I(x;\tau)$.

Let us also introduce the following metric characteristic (see [6])

$$m_f^M(x;\delta):=\sup\{\Omega^M(f,I(x;\tau)) : |\ln\tau|\leq\delta\}, x\in R_+, \delta\in R_+.$$

It is easy to see that the function $m_f^M(x;\delta)$ takes only non-negative values and is monotonically increasing in argument $\delta\in(0,+\infty)$.

Let be $\varphi(r)$ a non-negative, monotonically increasing $(0,+\infty)$ function. Let $MO_\varphi^M(x_0)$ denote the class of all functions $f\in L_{loc}(R_+)$ such that

$$m_f^M(x_0;r)=O(\varphi(r)),\ r>0.$$

Let $K\in L\left(R_+;\frac{dx}{x}\right)$ and

$$\int_0^\infty K(x)\frac{dx}{x}=1.$$

If we denote $K_\varepsilon(x)=\frac{1}{\varepsilon}K\left(x^{\frac{1}{\varepsilon}}\right),\ \varepsilon>0$, then we have

$$\int_0^\infty K_\varepsilon(x)\frac{dx}{x}=\frac{1}{\varepsilon}\int_0^\infty K\left(x^{\frac{1}{\varepsilon}}\right)\frac{dx}{x}=\int_0^\infty K(u)\frac{du}{u}=1.$$

A function of this type $K_\varepsilon(x)$ is called a Fejér-type Mellin kernel.

Let us consider the Melin singular integral with a Fejér-type kernel.

$$\Phi_{\varepsilon}(f;x)=\frac{1}{\varepsilon}\int_{0}^{\infty} f\left(\frac{x}{t}\right)K\left(t^{\frac{1}{\varepsilon}}\right)\frac{dt}{t},$$

where $f \in L_{loc}(R_+)$ is such that the integral exists almost everywhere in R_+. By changing the variable, it can be shown that

$$\Phi_{\varepsilon}(f;x)=\frac{1}{\varepsilon}\int_{0}^{\infty} f(u)K\left(\left(\frac{x}{u}\right)^{\frac{1}{\varepsilon}}\right)\frac{du}{u}=\int_{0}^{\infty} f\left(\frac{x}{u^{\varepsilon}}\right)K(u)\frac{du}{u}.$$

In particular, if $K(u)=\frac{1}{2}X_{I\left(1;\frac{1}{e}\right)}(u)$, where X_E is the characteristic function of the set E, then it can be shown that for this kernel

$$\Phi_{\varepsilon}(f;x)=f_{I(x;e^{-\varepsilon})}, \text{ Where } x\in R_+,\ \varepsilon>0.$$

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3. Theorem 1 .1.

Let $f \in L_{loc}(R_+)$, be K a Fejér-type kernel, $k(\tau):=\sup\{|K(t)|:|\ln t|\ge\tau\},\ \tau>0$, $k\in L(R_+), x_0\in R_+, \varepsilon>0$. Then the inequality holds true

$$\left|\Phi_{\varepsilon}(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right|\le c(k)\left(m_f^M(x_0;\varepsilon)+\int_{0}^{\infty}k(t)m_f^M(x_0;4\varepsilon t)dt+\right.$$

$$\left.+\int_{0}^{\varepsilon}\frac{m_f^M(x_0;t)}{t}\left(\int_{0}^{\frac{t}{4\varepsilon}}k(x)dx\right)dt+\int_{\varepsilon}^{\infty}\frac{m_f^M(x_0;t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^{\infty}k(x)dx\right)dt\right), \quad (1.1)$$

where $c(k)$ is a positive constant depending only on the function k.

Proof. Let $x_0=e^{-y_0}$ $(y_0\in R)$, $\Phi_{\varepsilon}^{*}(f;y_0)=\Phi_{\varepsilon}(f;e^{-y_0})=\Phi_{\varepsilon}(f;x_0)$. Then we have

$$\Phi_{\varepsilon}(f;x_0)=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}K^{*}\left(\frac{y_0-t}{\varepsilon}\right)f^{*}(t)dt, \varepsilon>0,$$

where $K^*(u):=K(e^{-u}),\ f^*(t)=f(e^{-t})$. Moreover, it is easy to see that

$$f_{I(x_0;e^{-\varepsilon})}=\frac{1}{2\varepsilon}\int_{y_0-\varepsilon}^{y_0+\varepsilon}f^{*}(t)dt=:f^{*}_{B(y_0;\varepsilon)}.$$

Taking into account the previous reasoning, we get that

$$\left|\Phi_{\varepsilon}(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right|\le\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\left|K^{*}\left(\frac{y_0-t}{\varepsilon}\right)\right|\left|f^{*}(t)-f^{*}_{B(y_0;\varepsilon)}\right|dt. \quad (1.2)$$

Due to the fact that $k(\tau)=\sup\{|K^*(y)|:|y|\geq\tau\},\tau\in(0,+\infty)$, from (2.2) we have

$$\left|\Phi_\varepsilon(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right|\leq\frac{1}{\varepsilon}\int_{-\infty}^{\infty}k\left(\left|\frac{y_0-t}{\varepsilon}\right|\right)\left|f^*(t)-f^*_{B(y_0;\varepsilon)}\right|dt,$$

Where $x_0=e^{-y_0}\ \ (x_0\in R_+,y_0\in R)$.

From the definition it is clear that is $k(\tau)$ a monotonically decreasing $(0,+\infty)$ function.

Next, we get that

$$\left|\Phi_\varepsilon(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right|\leq\sum_{n=-\infty}^{\infty}\frac{1}{\varepsilon}\int_{B(y_0;2^{n+1}\varepsilon)\setminus B(y_0;2^n\varepsilon)}k\left(\frac{|y_0-t|}{\varepsilon}\right)\left|f^*(t)-f^*_{B(y_0;\varepsilon)}\right|dt\leq$$

$$\leq\sum_{n=-\infty}^{\infty}\frac{1}{\varepsilon}\int_{2^n\varepsilon<|y_0-t|\leq2^{n+1}\varepsilon}k\left(\frac{|y_0-t|}{\varepsilon}\right)\left|f^*(t)-f^*_{B(y_0;2^{n+1}\varepsilon)}\right|dt+$$

$$+\sum_{n=-\infty}^{\infty}\left|f^*_{B(y_0;2^{n+1}\varepsilon)}-f^*_{B(y_0;\varepsilon)}\right|\int_{2^n\varepsilon<|y_0-t|\leq2^{n+1}\varepsilon}\frac{1}{\varepsilon}k\left(\frac{|y_0-t|}{\varepsilon}\right)dt=$$

$$=\sum_{n=-\infty}^{\infty}i_{1n}+\sum_{n=-\infty}^{\infty}i_{2n}\,.\ (1.3)$$

We will evaluate each of the terms on the right side of the relation (1.3).

If $n=0,\pm1,\ldots$, then we have

$$i_{1n}\leq\frac{1}{\varepsilon}k(2^n)\int_{2^n\varepsilon<|x-t|\leq2^{n+1}\varepsilon}\left|f^*(t)-f^*_{B(y_0;2^{n+1}\varepsilon)}\right|dt\leq$$

$$\leq2^{n+2}\cdot k(2^n)\cdot\frac{1}{2\cdot2^{n+1}\varepsilon}\int_{B(y_0;2^{n+1}\varepsilon)}\left|f^*(t)-f^*_{B(y_0;2^{n+1}\varepsilon)}\right|dt=$$

$$=2^{n+2}\cdot k(2^n)\cdot\Omega^M(f,I(x_0;e^{-2^{n+1}\varepsilon}))\leq2^{n+2}\cdot k(2^n)\cdot m_f^M(x_0;2^{n+1}\varepsilon)\,.\ (1.4)$$

Now let's consider the terms i_{2n}. If $n>-1$, then we have

$$i_{2n}\leq\frac{1}{\varepsilon}k(2^n)2^{n+1}\varepsilon\cdot\left|f^*_{B(y_0;2^{n+1}\varepsilon)}-f^*_{B(y_0;\varepsilon)}\right|=$$

$$=2^{n+1}\cdot k(2^n)\cdot\left|f_{I(x_0;e^{-2^{n+1}\varepsilon})}-f_{I(x_0;e^{-\varepsilon})}\right|.$$

From here, by virtue of Theorem 1.1, we obtain

$$i_{2n} \le 2^{n+1} \cdot k(2^n) \cdot \frac{2}{\ln 2}\left(m_f^M(x_0;2^{n+1}\varepsilon) + \int_{\varepsilon}^{2^{n+1}\varepsilon} \frac{m_f^M(x_0;t)}{t}dt \right). \quad (1.5)$$

If $n = -1$, then $i_{2n} = 0$.

Finally, let us consider the case $n < -1$. Then we have

$$i_{2n} \le \frac{1}{\varepsilon} k(2^n) 2^{n+1}\varepsilon \cdot \left| f^*_{B(y_0;\varepsilon)} - f^*_{B(y_0;2^{n+1}\varepsilon)} \right| =$$

$$= 2^{n+1} \cdot k(2^n) \cdot \left| f_{I(x_0;e^{-\varepsilon})} - f_{I(x_0;e^{-2^{n+1}\varepsilon})} \right| \le$$

$$\le 2^{n+1} \cdot k(2^n) \cdot \frac{2}{\ln 2}\left(m_f^M(x_0;\varepsilon) + \int_{\varepsilon}^{2^{n+1}\varepsilon} \frac{m_f^M(x_0;t)}{t}dt \right). \quad (1.6)$$

By virtue of the relations (1. 10) - (1 . 13) we obtain

$$\left| \Phi_\varepsilon(f;x_0) - f_{I(x_0;e^{-\varepsilon})} \right| \le \sum_{n=-\infty}^{\infty} 2^{n+2} \cdot k(2^n) \cdot m_f^M(x_0;2^{n+1}\varepsilon) +$$

$$+ \frac{1}{\ln 2} \sum_{n=-\infty}^{-2} 2^{n+2} \cdot k(2^n) \cdot m_f^M(x_0;\varepsilon) +$$

$$+ \frac{1}{\ln 2} \sum_{n=-\infty}^{-2} 2^{n+2} \cdot k(2^n) \cdot \int_{2^{n+1}\varepsilon}^{\varepsilon} \frac{m_f^M(x_0;t)}{t}dt +$$

$$+ \frac{1}{\ln 2} \sum_{n=0}^{\infty} 2^{n+2} \cdot k(2^n) \cdot m_f^M(x_0;2^{n+1}\varepsilon) +$$

$$+ \frac{1}{\ln 2} \sum_{n=0}^{\infty} 2^{n+2} \cdot k(2^n) \cdot \int_{\varepsilon}^{2^{n+1}\varepsilon} \frac{m_f^M(x_0;t)}{t}dt. \quad (1.7)$$

It can be shown that if $\varphi(x)$ a non-negative monotonically increasing function on an interval is true $(0,+\infty)$, then the following inequalities are true:

$$\sum_{n=-\infty}^{\infty} k(2^n) \cdot \varphi(2^{n+1}\varepsilon) \cdot 2^n \le 2 \cdot \int_0^{\infty} k(x)\varphi(4\varepsilon x)dx, \ \varepsilon > 0; \quad (1.8)$$

$$\sum_{n=-\infty}^{-2} 2^n \cdot k(2^n) \le 2 \cdot \int_0^{\frac{1}{4}} k(x)dx; \quad (1.9)$$

$$\sum_{n=0}^{\infty} k(2^n) \cdot \varphi(2^{n+1}\varepsilon) \cdot 2^n \le 2 \cdot \int_{\frac{1}{2}}^{\infty} k(x)\varphi(4\varepsilon x)dx, \ \varepsilon > 0; \quad (1.10)$$

Moreover, if $\psi(x)$ a non-negative monotonically decreasing function on an interval is true $(0,+\infty)$, then the inequality is true

$$\sum_{n=-\infty}^{-2} k(2^n)\cdot\psi(2^{n+1}\varepsilon)\cdot 2^n \le 2\cdot\int_0^{\frac{1}{4}} k(x)\psi(2\varepsilon x)dx,\ \varepsilon>0\,. \quad (1.11)$$

Using inequalities (1.15) -(1.18) from inequality (1.14) we obtain that

$$\left|\Phi_\varepsilon(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right| \le c(k)\left(m_f^M(x_0;\varepsilon)+\int_0^\infty k(x)m_f^M(x_0;4\varepsilon x)dx + \right.$$

$$\left. +\int_0^{\frac{1}{4}} k(x)\left(\int_{2\varepsilon x}^{\varepsilon}\frac{m_f^M(x_0;t)}{t}dt\right)dx+\int_{\frac{1}{2}}^{\infty} k(x)\left(\int_{\varepsilon}^{4\varepsilon x}\frac{m_f^M(x_0;t)}{t}dt\right)dx\right), \quad (1\ .12)$$

Where $c(k):=\dfrac{8}{\ln 2}\cdot\max\left\{2,\int_0^{\frac{1}{4}} k(x)dx\right\}$.

Changing the order of integration in the integrals of inequality (1.12), after some elementary transformations we obtain inequality (1.1). The theorem is proved.

4. Theorem 1.2

Let φ be a non-negative, monotonically increasing $(0,+\infty)$ function, K and k the same as in the previous theorem, and the following conditions are satisfied

$$1)\ \int_0^{\varepsilon}\frac{\varphi(t)}{t}dt = O(\varphi(\varepsilon)),\ \varepsilon>0\,;$$

$$2)\ \int_0^{\infty} k(x)\varphi(4\varepsilon x)dx = O(\varphi(\varepsilon)),\ \varepsilon>0\,;$$

$$3)\ \int_{\varepsilon}^{\infty}\frac{\varphi(t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^{\infty} k(x)dx\right)dt = O(\varphi(\varepsilon)),\ \varepsilon>0\,.$$

Then if $x_0\in R_+$ and $f\in MO_\varphi^M(x_0)$, then the inequality is true

$$\left|\Phi_\varepsilon(f;x_0)-d_f(x_0)\right|\le c\cdot\|f\|\cdot\varphi(\varepsilon),\ \varepsilon>0\,,$$

where $\|f\|:=\sup\left\{\dfrac{m_f^M(x_0;t)}{\varphi(t)}:t>0\right\}$, and is $c>0$ a constant depending only on the function k.

5. Conclusion

In this paper, we have investigated the asymptotic behavior of approximation Mellin singular integral operators in the class of summable functions. By analyzing the limiting process as the regularization parameter tends to zero, we have derived explicit expressions describing the asymptotic value of these operators. The obtained results not only generalize known theorems on singular integral approximations but also provide a more refined understanding of their convergence properties.

Our findings demonstrate that the asymptotic value depends essentially on the local behavior of the kernel near the singularity and on the summability properties of the underlying function. This insight allows for a more precise error estimation in practical approximation schemes and strengthens the theoretical foundation for the use of Mellin operators in applied problems.

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